

TRIGONOMETRY

FOR BEGINNERS



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BY

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PREFACE.

TRIGONOMETRY is of all the subjects which can be classed as Mathematics the most practical, and yet in most elementary text-books it is explained in a manner which is purely theoretical. Of late years the question has been raised whether it is not possible to teach symbolical mathematics in such a way that the student is from the beginning able to attach a practical meaning to every symbol which he uses.

Experience shows that a mathematical subject taught from the theoretical side is almost repulsive to all but the very few boys who have the very exceptional gift of a taste for symbols, while practical surveying and other allied subjects are attractive to the much larger class who are endowed with the practical sense. It seems a pity that the elements of trigonometry, which are quite simple when once they are grasped, should be rendered distasteful to those to whom they would be most useful, by the way in which they are generally presented. The method of prefacing the teaching of theoretical geometry by a course of geometrical drawing is no doubt a step in the right direction, provided too much time is not given to a subject which, from a mathematical point of view, is only preliminary.

It seemed to the writers of the following pages that it ought to be possible to apply a similar method to Trigonometry.

Accordingly, an attempt has been made to provide an

introduction to Trigonometry which, while it is thoroughly sound, is based on examples all of which can be realized in thought, while many of them can be actually worked out in practice.

Thus it will be seen that the writers have endeavoured to provide a text-book which, while it is introductory to Higher Mathematics, will also give to engineering students that knowledge of Trigonometrical principles which is absolutely necessary for them. The first 144 pages form a treatise on Trigonometry for beginners which may be considered as an introduction to any elementary mathematical text-book on the subject.

To this an Appendix is added, giving an explanation of Circular Measure, and of the extension of the definitions of the Trigonometrical Ratios to angles of any magnitude and the relations between the Ratios of angles whose sum or difference is any multiple of $\pi/2$. This renders the book a text-book suitable for students preparing for the Examination in Stage II. Mathematics of the Board of Education.

Explanations are given of the principles and construction of certain instruments, viz. the Level, the Vernier, Sextant and the Theodolite. It is hoped that students will in all cases be able to handle the instruments themselves when reading the description.

Considerable use is made of accurate drawing to scale, and explanations, based upon such drawings, are given of the theory of the Trigonometrical Ratios, of Tables of their values, of their rate of increase, and of the principles of interpolation. With regard to this principle, 'the Theory of Proportional Parts' (as it is called) has often been to students merely a *rule* by

which certain results can be obtained and nothing more; a good example of the mere mechanical manipulation of figures without any intelligent grasp of the meaning of the process. The use of graphs, introducing the idea of a continuous curve, provides an explanation which may possibly help to clear up this difficulty. In the Table of Logarithms on pages 88, 89 we have given to *five* figures the logs. of numbers up to 3 figures and have left it to the student to interpolate. In this way we attain sufficient accuracy for all practical calculations (an accuracy which is not attained by four figure tables) with great economy of space and at the same time provide very instructive exercises for the beginner.

The book has been written in the hope of providing an intelligible introduction to Trigonometry and not as a cram book for an examination, although the standard of the examination of Stage II. of the Board of Education has to some extent suggested the limits to be aimed at.

We have to thank the Controller of His Majesty's Stationery Office for permission to print examination papers in Trigonometry set by the Board of Education.

We have also to thank Messrs Griffin, London (Figs. 16, 65), Messrs Davis, Derby and London (Figs. 66, 67), Messrs Stanley, London (Fig. 72) for permission to use reproductions from their catalogues and instruments.

We shall be grateful for any suggestions for the improvement of the work, and for any note of errors or misprints which may be discovered by those using it.

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TRIGONOMETRY FOR BEGINNERS.

INTRODUCTORY.

§ 1. The Symmetry of a Circle.

1. Take a piece of thin tough paper (or tracing linen) and on it draw a circle with centre O and any convenient radius; draw a diameter AB .

Fold the paper carefully, figure outwards, about the diameter AB . Hold the folded paper up to the light.

It will be found that the two parts into which the circumference of the circle is divided by AB coincide with one another; that is, the two semicircles are congruent.

Shew, by folding the paper about other diameters, that any pair of semicircles are congruent.

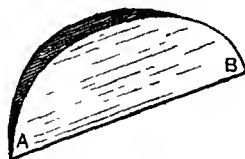


FIG. 1.

NOTE. When a figure has been drawn on paper, and the paper is to be folded about some line connected with the figure, it is usual to say "Fold the figure about the straight line." A similar meaning is attached to the instructions, "Cut out the figure," "Place the triangle upon the other."

2. Draw two circles with equal radii, say 1.5 inches; cut the circles out and, by passing a pin through the centre of each, place them so that their centres coincide. What do you observe?

Can you say that any two concentric circles of equal radii are congruent?

3. Draw a circle with centre O and any convenient radius; draw a diameter AB . Mark with a pinhole the position of any point on the circumference.

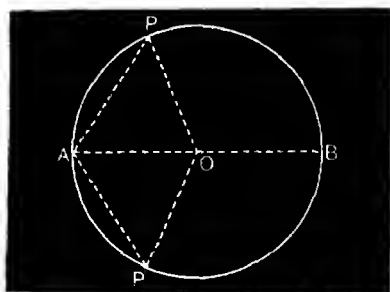


FIG. 2.

Fold the circle about AB , and with the pin mark the point (P') upon which P falls. Open the paper out flat and join OP , OP' . [Fig. 2.]

Is P' on the circumference of the circle? [Why?]

Is $\angle AOP = \angle AOP'$? [Why?]

Is chord $AP =$ chord AP' ? [Why?]

Is arc $AP =$ arc AP' ? [Why?]

Points, such as P and P' in Experiment 3, and straight lines, which coincide when a figure is folded about a straight line, are called **corresponding points** and **corresponding straight lines**.

Thus, if $ABCD$ is a square, B and D are corresponding points with regard to the diagonal AC , and AB , AD and CB , CD are corresponding lines with regard to this diagonal.

4. Fold the circle used in Expt. 3 about the line OP ; mark the points (Q, R) upon which A, P' fall respectively.

Are Q, R on the circumference? [Why?]

Is $\angle POQ = \angle QOR$? [Why?]

Is chord $PQ =$ chord QR ? [Why?]

Is arc $PQ =$ arc QR ? [Why?]

If a plane figure can be folded about a straight line so that the two parts, into which the figure is divided by the straight line, coincide, the figure is said to be **symmetrical with regard to the straight line**; the straight line is called an **axis of symmetry** of the figure, and any point or line in one of the parts is called the **image** or **reflection in the straight line** of the corresponding point or line in the other part.

Hence, it follows from Experiment 1, that a circle is **symmetrical about any diameter**. In Fig. 2, P' is the **image** (or reflection) of P in the diameter AB .

5. Draw a circle, and fold it about any chord which is not a diameter.

Is a circle symmetrical about any other line except a diameter?

6. Mark on a piece of paper a circle by running the point of a pencil round the edge of a penny.

Cut out the circle, and, by folding, crease the paper along axes of symmetry. It will be found that everyone of these axes passes through one point (the centre of the circle).

Hence, it follows that

- (1) Every diameter of a circle is an axis of symmetry;
- (2) Every axis of symmetry of a circle is a diameter.

7. Take any three points P, Q, R on a sheet of paper. Holding the paper up to the light, fold it (i) so that P and Q coincide, (ii) so that Q and R coincide, marking the crease in each case. Shew that the point in which the creases intersect is the centre of the circle passing through P, Q, R .

It follows from Expt. 7 that, if two circles have three points in common, they have the same point as centre, and therefore, by Expt. 2, are congruent.

Hence **two different circles cannot intersect in more than two points.**

Again two intersecting circles have different centres; and therefore the line passing through these centres is the only line which is a diameter of both circles; that is, **two intersecting circles have only one axis of symmetry, which is the line joining their centres.**

8. Draw a circle with centre O and any radius.

Take any point A on the circumference and with A as centre, and unequal radii, draw two arcs of circles cutting the first circle in P, P' and Q, Q' respectively.

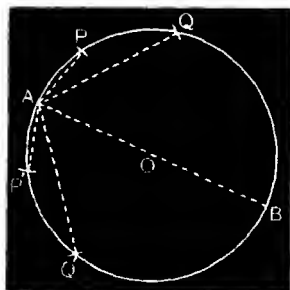


FIG. 3.

Then $AP = AP'$, and $AQ = AQ'$. [Why?]

In what line are P', Q' the images of P, Q ? [Why?]

Shew that $\text{arc } PQ = \text{arc } P'Q'$ and $\angle POQ = \angle P'OQ'$.

9. Verify, by folding, that if P, Q, P', Q' are four points on the circumference of a circle whose centre is O , and

(i) if $\angle POQ = \angle P'OQ'$; then chord $PQ =$ chord $P'Q'$, and arc $PQ =$ arc $P'Q'$;

(ii) if arc $PQ =$ arc $P'Q'$; then chord $PQ =$ chord $P'Q'$, and $\angle POQ = \angle P'OQ'$;

(iii) if chord $PQ =$ chord $P'Q'$; then arc $PQ =$ arc $P'Q'$, and $\angle POQ = \angle P'OQ'$.

The results obtained in Expt. 9 are very important. By means of them we are enabled to devise instruments and methods for measuring and reproducing angles.

§ 2. Protractors.

In order to measure numerical values of angles, a unit angle must first be decided upon.

In Geometry the unit used is the **right angle**. It is convenient for the following reasons:—

- (1) Its shape is familiar,
- (2) Its definition is simple,
- (3) It can easily be constructed,
- (4) It is constant, for it can be shewn that all right angles are equal.

In Practical Trigonometry the unit chosen is the ninetieth part of a right angle; this angle is called a **degree**.

If the arc of a semicircle is divided into 180 equal parts, and the points of division are joined to the centre, there will be 180 equal angles at O . Hence, since there are 180 degrees in two right angles, each of these equal angles is a **degree**.

The degree is divided into 60 minutes, and the minute into 60 seconds.

The signs $^{\circ}$, $'$, $''$ are used as abbreviations for "degrees," "minutes," "seconds." Thus we write $6^{\circ} 23' 46''$ for "six degrees + twenty-three minutes + forty-six seconds."

We have therefore

$$1 \text{ rt. } \angle = 90^{\circ},$$

$$1^{\circ} = 60',$$

$$1' = 60''.$$

The arc of a circle subtending at its centre an angle of 1° is only slightly greater than $\frac{1}{60}$ th of the radius; so that it is impossible without using an inconveniently large radius to make a diagram shewing arcs corresponding to much smaller angles than 1° .

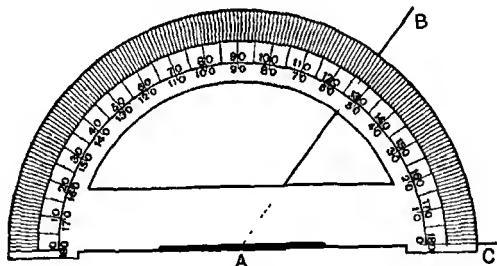


FIG. 4.

The usual form of the **circular protractor** (Fig. 4), in which the radius of the outer graduated semicircular arc is about 1.6 of an inch, reads to degrees, and by estimation to half a degree, the divisions being about .03 of an inch apart.

A useful form is **Low's 4-inch quadrantal protractor** reading to half-degrees between 0° and 90° . It is made

of thin white celluloid; and angles may be estimated with it to a tenth of a degree with practice.

Circular protractors made of thin transparent celluloid, with the divisions marked on the underside, are also sold, and these are extremely handy and accurate.

10. Measure a given angle BAC [Fig. 4].

Place the protractor so that the vertex of the angle is just visible in the notch marking the centre of the semicircular arcs and one of the arms of the angle is in a direct line with the division marked 0° . then read off the number of degrees corresponding to the division which is in a direct line with the other arm of the angle.

Thus in Fig. 4, the angle BAC is 53° (approx.).

The circular protractor may also be used to draw an angle equal to a given angle.

11. Draw an angle of 53° with a circular protractor.

Method (a). In Fig. 4. Suppose AC to be one of the arms of the angle to be drawn: place the protractor so that the central notch is at A, and bring the division marked 0° into a direct line with AC; draw a *short straight line* in a direct line with the line joining A to the point on the arc corresponding to 53° : remove the protractor and draw the arm AB passing through A and the mark you have made.

Method (b). Suppose AB is the arm of the angle first drawn: place the protractor so that the central notch is at A and bring the point on the arc corresponding to 53° directly over the line AB: then, using the diametral edge of the protractor as a ruler, draw the other arm AC. [It must be noticed that for this method the little feet of the protractor in Fig. 4 should be absent, i.e., the straight edge of the protractor must coincide with the division marked 0° .]

The method (b) is the principle of an improved protractor designed by Professor Low [Fig. 5]. It consists of two parts made of white celluloid; a segmental part DE, the circular edge of which has a tongue which fits into a corresponding groove on the right-angled part as shewn. The circular edge of DE is graduated in degrees, and a mark on the circular edge of ABC, shown in Fig. 5, in coincidence with 45° on DE; so that the straight or drawing edge of DE may be set to any angle from 0° to 90° with AB or BC. The effective diameter of the protractor is 7 inches



FIG. 5

12. On a piece of tough stout paper or thin cardboard draw a rectangle, 6 in. long and $1\frac{1}{2}$ in. broad, and mark the middle point of one of the longer sides. Place the circular protractor with its diametral edge along this side, the notch being at the middle point.

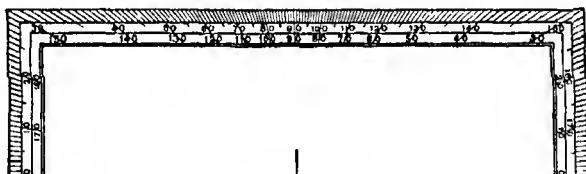


FIG. 6.

Mark off with a needle-point, or a well-sharpened hard pencil, the divisions on the semicircular arc from 0° to 180° . Remove the protractor, and join each of the marks to the mark made at the middle point of the side of the rectangle; draw four straight lines parallel to the other three sides of the rectangle and ink in, as shewn in Fig. 6.

A **rectangular protractor** has been constructed; it may be cut out and kept for use.

For constructing angles of given magnitude the rectangular protractor should be used as in method (b) for the circular protractor.

13. Construct, with the rectangular protractor, an angle of 37° .

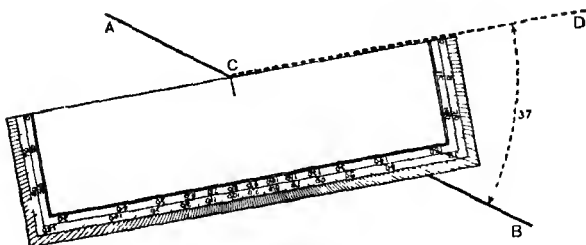


FIG 7.

Take a straight line AB , and C any point in it. Place the protractor so that the notch is at C , and the division for 37° is in a direct line with CB . Then using the edge of the protractor as a ruler draw CD the other arm of the angle BCD . Then BCD is 37° .

Note 1. The usual method involves the following operations —(i) placing the notch at a point, (ii) placing the edge in coincidence with a straight line, (iii) marking a point opposite a division, and (iv) joining two points. The operations (iii) and (iv) are the most liable to error and these are eliminated in the above method, which is therefore quicker and more accurate.

Note 2. The rectangular protractor can be looked upon as a circular protractor of varying radius, the divisions being further apart for angles from 0° to 45° , but nearer together from 45° to 90° . It should always be very carefully tested before being used for any drawing requiring any great accuracy.

If a given angle, drawn on a sheet of paper, has to be copied upon another sheet of paper it can be "pricked off" either (a) with three points, one of which is at the vertex of the angle, or more accurately, (b) with four points, two on each arm of the angle.

The sheet on which the angle is drawn should be laid perfectly flat upon the other, and the "pricker" should be held vertically whilst it is pushed through both sheets. If there is not a pricker amongst the drawing instruments, one may easily be made by mounting a fine needle in a handle. Frequently, however, on unscrewing the handle of the drawing-pen in a box of instruments, a needle will be found mounted in the handle.

If the given angle is drawn on material that is not thin or cannot be pierced, or if the copy has to be drawn on the same surface or in some special position, the ordinary Geometrical construction, depending on the property found in Expt. 8 (iii), must be used.

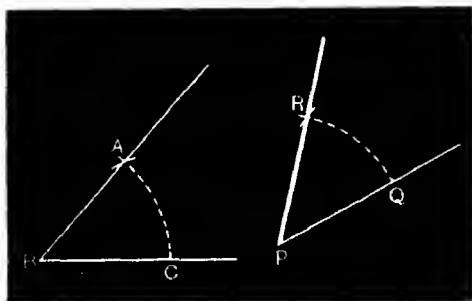


FIG. 8.

Thus, in Fig. 8, if ABC is the given angle, and an angle equal to it has to be drawn on the upper side of PQ , with its vertex at P , we proceed as follows:—With centre B and any radius describe a circle cutting the arms of the angle in C, A : with centre P and the same radius draw an arc QR : with centre Q and radius equal to *chord* CA , draw an arc intersecting the arc QR in R : join PR .

Then $\angle RPQ = \angle ABC$. [Why?]

If we decide always to use a constant radius for the first arc, the radius for the second arc (i.e. the *chord* subtending the given angle), will also be constant for any

given angle. Hence, these chords can be determined once for all from a circular protractor and a **scale of chords** made. This scale, if of sufficiently large "radius," can be used for measuring and constructing angles with great accuracy: being in the form of a scale it is also more handy than either form of protractor; it should always be used when great accuracy is necessary.

14. Draw OA, OB, two straight lines at right angles. With centre O and radius=4" describe an arc cutting the straight lines in A and B.

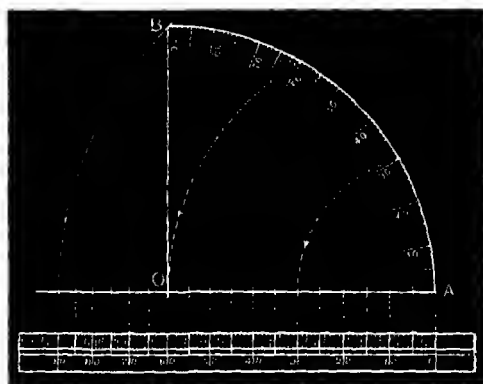


FIG. 9.

Graduate the arc in degrees with the circular protractor. With centre A and radii equal to the chords corresponding to $1^\circ, 2^\circ \dots 90^\circ$ respectively, describe arcs cutting AO. A **scale of chords** has been constructed, which may be inked in and figured, as in Fig. 9, and out for use.

Note. The lengths of the chords may be tabulated and a **table of chords** formed, which is fairly accurate and handy in use.

EXERCISES.

1. Given a scale of chords, how do you find the radius of the circle from which it has been constructed?
2. How many degrees are there in the angle between the hands of a watch at 4 o'clock?
3. How many degrees does (i) the hour hand, (ii) the minute hand rotate through in one minute?
4. What is the angle between the hands of a watch at (i) 4.25; (ii) 5.40; (iii) 9.7?
5. Draw six triangles of different shapes and sizes, and measure with the *circular protractor* the sum of the three angles in each triangle.
6. Draw four quadrilaterals of different shapes and sizes, and find by means of the *scale of chords* the sum of the four angles of each.
7. Express $\frac{4}{7}$ of a right angle in degrees, minutes and seconds.
8. How often does the angle $59^{\circ}44'42''$ contain the angle $2^{\circ}50'42''$?
9. One angle of a triangle is $50^{\circ}0'27''$; if one of the other angles is double of the third, find the other two angles.

Note. The sum of the interior angles of any rectilinear figure of n sides is $2n - 4$ right angles.

10. Using either protractor or scale of chords, draw regular polygons of 4, 5, 6, 8, and 10 sides.

§ 3. Points of the Compass.

Sailors, in reckoning the direction of a ship's motion, use an eighth of a right angle as unit angle; this angle is called a **point**. For sub-divisions they use **half-points** and **quarter-points**.

A Mariner's Compass Card is shewn in Fig. 10.



FIG. 10.

The names of the points are derived from those of the four cardinal points as follows:

N.W. (Nor'-West) bisects the angle between North and West. S.S.E. (Sou'-Sou'-East) bisects the angle between South and South-East. "By" is short for "altered by"; thus S.by E. indicates a direction South altered by *one point* towards the East; W.N.W. by $\frac{3}{4}$ W., a direction bisecting the angle between West and North-West altered by *three-quarters of a point* towards the West, and is the same as W. by N. by $\frac{1}{4}$ N., making an angle of about 14° with West.

1. How many degrees are there in a point?
2. Find the angle between the directions, (i) W.by N. and N.N.W., (ii) S.E.by E. and E.N.E.
3. Draw a straight line ON, where N is north of O, and make a diagram shewing the positions of objects whose distances and directions from O are given as:—

B, 2 miles, E.by S. by $\frac{3}{4}$ S.; C, 3 miles, S.S.E.; D, 4 miles, S.W. by $\frac{3}{4}$ S.; E, $2\frac{1}{2}$ miles, W. by N.; F, $3\frac{1}{2}$ miles, N.N.W. by $\frac{1}{4}$ W.

Measure the angles of the polygon BCDEF.

The direction in which an object lies, as viewed from a place in the same horizontal plane, is called its **bearing**. If the bearing of an object cannot be exactly expressed by points of the compass, degrees are used.

Thus, in Fig. 11, the bearing of B from O can be expressed either as

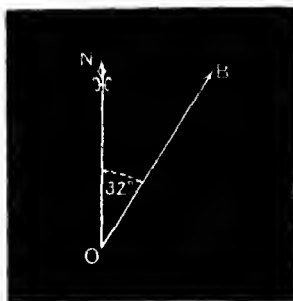


FIG. 11.

32° E. of N., 58° N. of E.,

or

N. by 32° E., E. by 58° N.

4. The bearing of O from B is S. by 32° W. or W. by 58° S
[Draw a figure to prove this statement.]

5. Find the distance and bearing of each of the points C, D, E, F from B, in Ex. 3, p. 13.

6. As I steer a ship, I see a red, and a white, light, which I know (from a chart) proceed from two lighthouses, the one showing the red light being 10 miles due north of the other. The red light bears N. by 25° W., the white W. by 23° N. Draw a figure representing the position of the ship and measure its distance from the red light.

7. A ship steaming N.N.W. at 18 knots an hour, is due west of a lighthouse at 2 a.m.; the light bears S.E. at 2.37 a.m. Draw a figure, shewing the lighthouse and the track of the ship; and measure the distance and bearing of the lighthouse from the ship at 3 a.m.

8. As I walked along a straight road in a direction E.N.E., I noticed two churches A and B, bearing N.N.E., and N.E. by N., respectively. These churches came into line 300 yds. further along; at the end of another 200 yds. the bearing of B was W. by 25° N., whilst A was hidden by a hill. Find the distance of A from B.

In general practical work it is not necessary to find the bearing of a single object, but the difference in bearing of two objects, i.e. the angle subtended by the objects (or the straight line joining them) at the observer's eye, if the objects and the observer's eye are all at the same level.

If this is not the case, the difference of bearing is the angle subtended by the projections of the objects upon a horizontal plane through the eye.

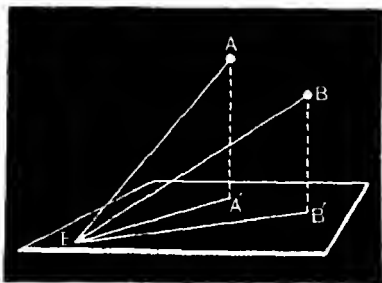


FIG. 12.

Thus, in Fig. 12, let A and B represent two objects; let AA' , BB' be vertical (as ascertained by a plumb-line); let E be the observer's eye, and EA' , EB' perpendicular to AA' , BB' : then EA' , EB' are horizontal lines, and the plane containing EA' , EB' is a horizontal plane. The difference of bearing of A , B as seen from E is $\angle A'EB'$, whilst the angle subtended by A , B at E is $\angle AEB$, which is *not equal* to $\angle A'EB'$ as a general rule.

The angle AEA' is called the **elevation** or (angular) **altitude** of A as seen from E .

If a horizontal line AC is drawn, the angle CAE is called the **depression** of E as seen from A .

9. If P , Q are two objects, shew that the elevation of P viewed from Q is equal to the depression of Q viewed from P .

§ 4. The Level and the Theodolite.

A level is used to ascertain whether a line is horizontal.

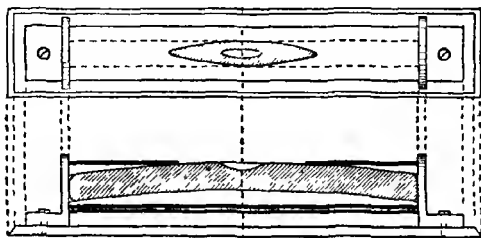


FIG. 13.

Fig. 13 shews the plan and sectional elevation of an ordinary* "carpenter's level." It consists of a closed circular cylindrical tube, nearly full of spirit, fitted into a wooden frame with a plane metal base. The tube, not being quite full, has a bubble of air left in it which occupies the highest point in the tube. The instrument is so constructed that when the bubble is bisected by a mark on the tube, seen through a little glass window in the top of the instrument, the base lies along a horizontal line in the direction of its length.

To ascertain whether a plane surface of a given object is horizontal, two observations of the position of the bubble when the base of the level is applied to the surface in different directions, are *necessary* and *sufficient*. [Why?]

* The figure shews an improved form in which the frame is all metal, and has a revolving tube, which closes and protects the glass tube when not in use.

Another form of level is commonly used on Physical instruments and photographic cameras. It consists of

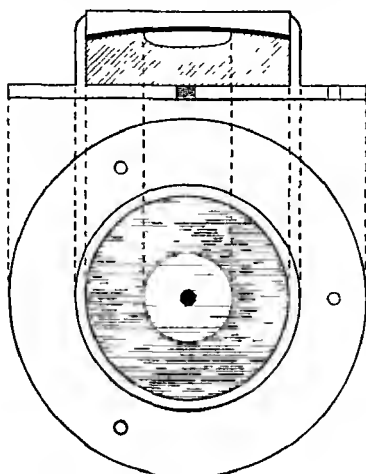


FIG. 14.

a shallow cylindrical metal box, into which is fitted a glass cover with its lower surface spherical. The box is *nearly* filled with spirit, leaving room for a bubble of air which, as in the ordinary level, occupies the highest point. It is so arranged that, when the base of the level is accurately horizontal, the centre of the bubble coincides with the centre of the glass top of the instrument.

This form is extremely handy; for, by one application of the instrument to a plane surface, we can ascertain whether that surface is horizontal. On physical instruments and cameras a level of this form is often permanently fixed; the position of the bubble then indicates how the instrument has to be tilted to make its base horizontal; for the displacement of the bubble from its

central position is upwards along the line of greatest inclination to the horizon of the surface to which the level is fixed.

15. Take a piece of wood, $6'' \times \frac{3}{4}'' \times \frac{3}{4}''$, accurately rectangular, and fit a piece of stout cardboard or thin wood, $8'' \times 6''$, into a slit in it, so that the cardboard is accurately perpendicular to one of the long faces *ED* of the wood. In the central line of the cardboard $3''$ from the top, mark a point *A*; with *A* as centre and $3''$ radius describe a circle and draw a diameter *CAB* through *A* parallel to the face of the wood *ED*. Graduate the arc in degrees, the lines *AB*, *AC* indicating 0° , and attach to the cardboard at *A*, by means of a long drawing pin, a cardboard pointer with two small drawing pins inserted, points outwards, near the ends (*G*).

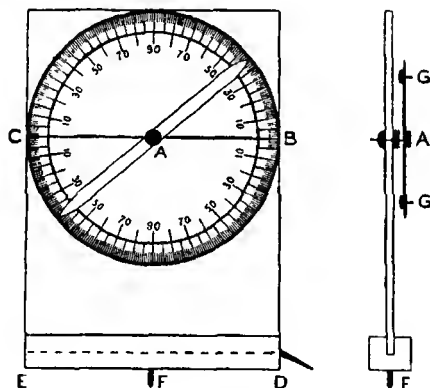


FIG. 15.

Use this instrument to obtain the elevation of objects in the schoolroom, such as the clock, the tops of the windows, placing it on a table or other plane surface, which has been "levelled."

What "horizontal" are you using?

16. Fasten a small spike in the wood (*F* in Fig. 15), directly under *A*, and fix a needle (*D* in Fig. 15) at one end of the wood so that it is in the plane of the cardboard but inclined to *CD*. On a piece of wood about $5''$ square, $\frac{1}{2}''$ thick, draw a

circle of 5" radius and graduate the circumference in degrees; insert the spike F at the centre of this circle. You have now constructed an instrument by means of which you can simultaneously determine the difference of bearing and the angular elevations of two objects.

This is a simple model of an important instrument used in Surveying, called a theodolite. Where possible

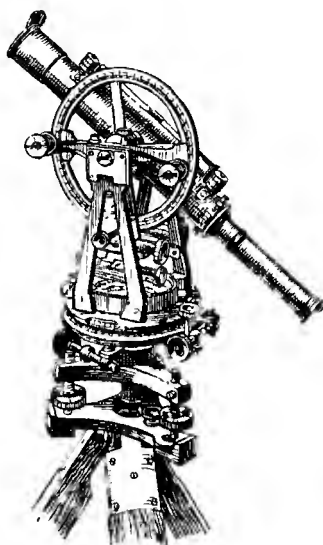


FIG. 16. The Theodolite*.

each student should make one of these models for himself in the school workshop, and "elementary practical surveying" should be practised with it in the playground, a circular level being fitted to the base board and a camera tripod used for a stand. A full description of the theodolite will be found on pages 127-130.

* From an instrument supplied by Messrs Griffin, London. This firm also supplies a reliable model theodolite for school use, which can be recommended.

§ 5. Graphical Solution of "Heights and Distances."

The sun is so far away from the earth that the "paths" of all those rays of sunlight which at any instant fall on different points within a small area are approximately parallel; that is, they all make the same angle with a horizontal plane.

This angle is called the **altitude of the sun** at that instant.

17. Set up a stump in the sunlight on a cricket pitch, being careful to get the stump perfectly vertical. [How?]



FIG. 17.

Measure the length of the part of the stump above the ground and of its shadow on the ground. Make a diagram ABC , in which B is a right angle and AB , BC represent the stump and its shadow, drawn to some convenient scale.

Measure the angle ABC ; i.e. find the altitude of the sun at the instant.

1. If the Sun's Altitude is 42° , what is the length of the shadow cast by a walking-stick 3 ft. 2 ins. long?
2. If a stick a yard long casts a shadow 25 inches long, what is the Sun's Altitude? [R. r. = $\frac{11}{16}$.]
3. The shadow of a poplar tree is 140 feet long; the shadow of a yard stick at the same time is 43 inches: find the height of the tree.

In question 3 above, a poplar tree was taken because such a tree grows nearly vertically. For, in finding the height of objects by a simultaneous determination of the length of its shadow and of the altitude of the sun, it is necessary to be able to measure the length of the shadow from its extremity to a point *vertically underneath the highest point of the object*, the measurement being made on a plane surface, or on *the same slope* as the measurements for determining the altitude of the sun.

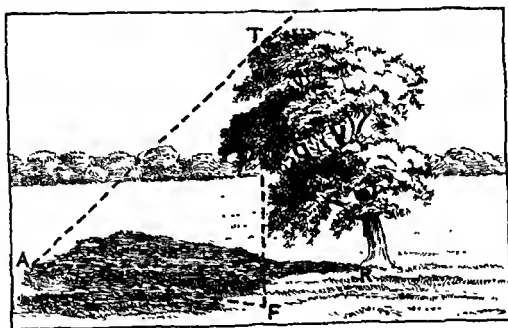


FIG. 18.

For instance, the length it would be necessary to measure in determining the height of the leaning tree in Fig. 18 is AF, where F is vertically under the highest point of the tree, (and not AR).

It would often be difficult to find accurately the *highest* point. In the figure as drawn T is not the highest point. It is the top of the tree as seen from A.

Again, in measuring the height of a hill it is, in general, impossible to get at the foot of the vertical line through the highest point.

Moreover the shadow will not usually fall conveniently for the purpose of measurement on a horizontal plane, so the elevation of the top of the object must be found at a convenient *station*, irrespective of the sun's altitude, by means of a theodolite.

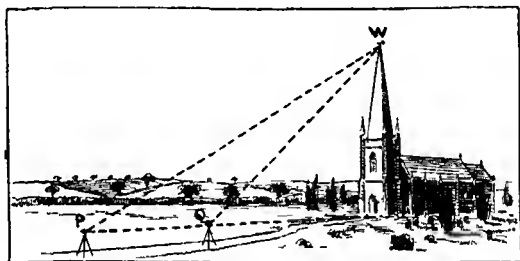


FIG. 19.

For example, to find the height of a weathercock W , we could by means of our model theodolite find the elevation of W at a station P , and, proceeding *straight towards* the vertical through W for a ft, find the elevation of W at Q . If P and Q are in a horizontal line or inclined to the horizontal plane at a known angle, then the angles P and Q and the distance PQ are sufficient data to draw the triangle PQW : from which the height can be found.

Even on very rough ground it may be possible to find isolated positions, here and there, which are in the same horizontal plane as the foot of the object whose height we wish to find. If two of these happen to be in a direct line with the object and their distance apart can be determined, then the height of the object can at once be found.

18. Fig. 20 (a) represents a hill whose height above a certain lake is required.

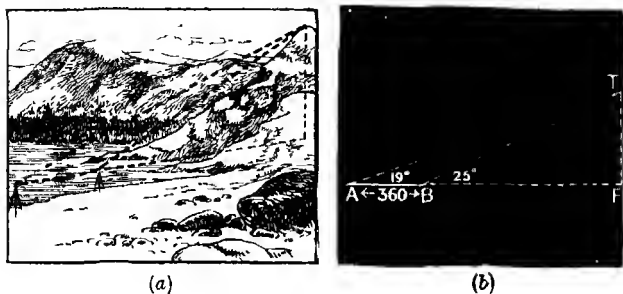


FIG. 20.

Two convenient stations A and B are found on the margin of the lake. AB is measured and the elevations of T, as seen at A and B, are observed. It is found that $AB = 120$ yds., elevation at A = 19° , and elevation at B = 25° : hence in the triangle ABT the side AB, the interior angle at A and the exterior angle at B are known, and the triangle can be constructed as follows:—

Draw a st. line ABF. Mark off $AB = 1.8$ in. [R.F. = $\frac{1}{2400}$]; make $\angle FAT = 19^\circ$ and $\angle FBT = 25^\circ$; draw $TF \perp AF$; measure TF.

It will be found that $TF = 2.38$ in.

$$\therefore \text{Height of the hill} = 2.38 \times 2400 \text{ ft.} \\ = 476 \text{ ft.}$$

Note.—This height is the height of T above the horizontal line through the pivot (A in the figure of the model theodolite of Expt. 16) of the telescope of the theodolite. The height of this above the level of the lake must be added to obtain the height of the hill above the level of the lake.

It is obvious that, if the distance between A and B is known (say they are consecutive milestones on a straight horizontal road), the angle of depression of each may be observed from T instead of the angles of elevation of T from A and B.

4. From the top of a cliff 130 feet high the angles of depression of two boats are 57° and 49° : if they are in a direct line with the foot of the cliff, find the distance between them. [Draw a sketch as well as the diagram to scale.]

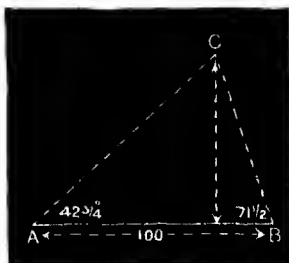
5. A flagstaff on top of a cliff, and at its edge, is 35 feet high. From a ship at sea the angles of elevation of the top and bottom of the pole are $16^\circ 30'$ and 18° respectively; find the distance of the ship from the cliff and the height of the cliff. [Draw a sketch as well as the diagram to scale.]

6. From the top of a lighthouse the depression of a ship at sea is 42° : from a window 23 feet lower down the depression is 36° . Find the distance of the vessel from the foot of the lighthouse, and the height of the lighthouse.

19. Fig. 21 (a) represents a river, whose width is required.



(a)



(b)

FIG. 21.

Two positions A and B on one bank are chosen and from the position A the difference in bearing of an object C, on the opposite bank, and of the other position B is observed: from B the difference in bearing of C and A is observed; AB is measured; i.e. AB and the angles at A and B of the triangle ABC are known and the perpendicular distance of C from AB can be found.

Find the breadth of the river given that $AB = 100$ yds., $\angle A = 42\frac{3}{4}^\circ$, $\angle B = 71\frac{1}{2}^\circ$. [R.F. = $\frac{1}{800}$.]

7. As I walk along a straight road, I notice a church whose direction makes an angle of 18° with the road; one mile further on its direction makes an angle of 72° ; what is the shortest distance of the church from the road?

8. If the spire of the church in question 7 is 200 ft. high, find its elevation at the two positions at which the church was observed.

9. A sphere of lead, one inch in diameter, is attached to a thread thirty inches long to form a pendulum: the pendulum vibrates through 10° on each side of the vertical. Find the greatest height to which the centre of the lead sphere rises, and the greatest displacement horizontally, assuming that the centre of the sphere always lies in a straight line with the string.

10. A fire-escape cannot get nearer the wall of a burning house than 15 feet. How long must the escape be to reach a window-sill 65 feet above the ground? What angle will the escape make with the ground?

11. A person, 5 ft. 10 in. high, walks under and past a street lamp fixed to a wall. When he is 28 feet from the point vertically under the lamp, his shadow is 20 ft. long: find the height of the lamp.

12. A telegraph pole, 50 ft. high, is prevented from bending by a wire attached to ring-bolts at the top and bottom of the pole and a horizontal stretching-rod 4 ft. long fixed to the pole at a height of 35 ft. Find the length of wire between the rings.

13. A flagstaff is fixed upright by four ropes attached to the top of the staff and to pegs in the ground 10 feet from the foot of the staff. If the flagstaff is 28 feet high, how much rope is required, allowing a foot at each end for tying?

14. A bridge is thrown across a ravine through which a stream 17 feet wide runs: the sides of the ravine slope upwards, right from the banks of the stream, at angles of 80° and 74° respectively. The bridge is 85 feet long. What is its height above the stream?

§6. Definitions of the Trigonometrical Ratios, Sine, Cosine and Tangent, for an acute angle.

20. Draw any acute angle BAC.

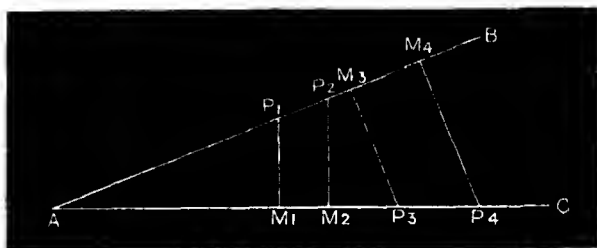


FIG. 22.

Take any number of points $M_1 M_2 M_3 \dots$, in either arm, and draw $M_1 P_1, M_2 P_2, M_3 P_3 \dots$ perpendicular to $AM_1, AM_2, AM_3 \dots$, as in Fig. 22. Measure the lengths MP, AM, AP, and insert them in a table, as below.

| | MP | AM | AP | $\frac{MP}{AP}$ | $\frac{AM}{AP}$ | $\frac{MP}{AM}$ |
|---|----|----|----|-----------------|-----------------|-----------------|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |

Work out the ratios $\frac{MP}{AP}, \frac{AM}{AP}, \frac{MP}{AM}$ as *decimals*, and insert them in the last three columns respectively.

If the measurements are carefully made, the numbers in each of the last three columns will be very nearly equal.

NOTE. Do not take the point P too near A, or else the percentage error in measuring the lengths of the lines will be large.

DEFINITIONS OF TRIGONOMETRICAL RATIOS 27

The same result will be obtained for any acute angle; this fact can be expressed thus:

If $\angle BAC$ is any given acute angle, M any point in either arm and MP at right angles to that arm, the values of each of the fractions

$$\frac{MP}{AP}, \quad \frac{AM}{AP}, \quad \frac{MP}{AM}$$

are constant for all positions of P .

These fractions, or ratios, are called respectively the *sine*, *cosine* and *tangent* of the angle A : the abbreviations $\sin A$, $\cos A$, $\tan A$ being used.

$$\text{Thus} \quad \sin A = \frac{MP}{AP}.$$

$$\cos A = \frac{AM}{AP},$$

$$\tan A = \frac{MP}{AM} = \frac{MP}{AM}$$

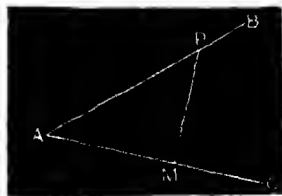


FIG. 23.

or in terms of the angles of the *right-angled* triangle formed from a given angle,

$$\sin(\text{angle}) = \frac{\text{side opposite angle}}{\text{side opposite the right angle}},$$

$$\cos(\text{angle}) = \frac{\text{side opposite other acute angle}}{\text{side opposite the right angle}},$$

$$\tan(\text{angle}) = \frac{\text{side opposite angle}}{\text{side opposite other acute angle}}.$$

If in Fig. 23 the angle APM is considered as the given angle P , and M as any point in one arm and MA as drawn at right angles to that arm, then

$$\sin P = \frac{\text{side opposite } P}{\text{side opposite } M} = \cos A,$$

$$\cos P = \frac{\text{side opposite } A}{\text{side opposite } M} = \sin A,$$

$$\tan P = \frac{\text{side opposite } P}{\text{side opposite } A} = \frac{1}{\tan A}.$$

The angles A and P , whose sum is a right angle, are called **complementary angles**.

Hence

$$\sin (\text{any angle}) = \cos (\text{complementary angle}),$$

$$\frac{1}{\tan (\text{any angle})} = \tan (\text{complementary angle}).$$

21. Draw a quadrant of a circle with a radius 10 in. Draw two radii OX , OY at right angles, and graduate the arc XY in degrees. Through each point of division (P) draw a perpendicular (PM) to OX . Measure these perpendiculars true to '01 in., divide each by 10 and thus construct a table of sines true to three places of decimals.

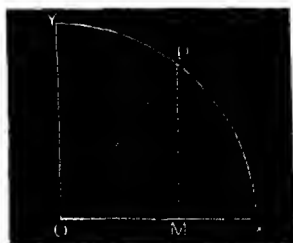


FIG. 24.

Compare the results with the tables on pp. 36, 37.

Construct from the same diagram a table of cosines and verify $\sin XOP = \cos (90^\circ - XOP)$.

22. Draw a quadrant of a circle with a radius 10 in. Draw two radii OX , OY at right angles, and graduate the arc XY in degrees. Through these points of division (P), produce the

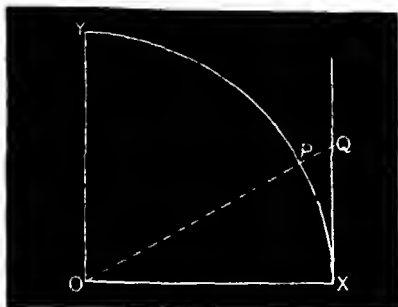


FIG. 25.

radius (OP) to meet the tangent at X in a point (Q). Measure the distance of each of these points from X , true to .01 in., divide by 10 and thus construct a table of tangents true to three places of decimals.

Compare the results with the tables on pp. 38, 39.

It will be found that as the angle XOP approaches 90° the value of the tangent is very large and increases as the angle increases. The student should verify, with the values obtained for angles between 45° and 60° and 45° and 30° , that the formula

$$\tan A \times \tan (90^\circ - A) = 1$$

is true; and, *assuming* this universally true between 0° and 90° (cf. p. 28), obtain the tangents for angles between 60° and 90° from the values observed for those between 30° and 0° . [Use decimal approximation to three places.]

1. Shew that $\tan A = \sin A \div \cos A$.

2. Let ABC be a triangle, right-angled at C ; produce BC to D , making $CD=BC$; join AD , and draw BE perpendicular to AD .

Take $AB=1$, and $\angle BAC=A$;
shew that

$$BE=\sin 2A$$

$$BD=2 \sin A,$$

and hence, in general, when A is
an acute angle less than 45° , that

$$\sin 2A \neq 2 \sin A.$$

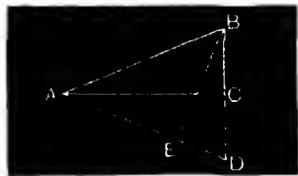


FIG. 26.

3. Let ABC be a triangle, right-angled at C ; produce AC to D , making $BD=AC$; join BD , and draw DE perpendicular to AB produced, meeting it in E .

Take $AB=1$, and $\angle BAC=A$;
shew that

$$BE=\cos 2A,$$

$$AD=2 \cos A,$$

and hence, when A is any acute
angle less than 45° , that

$$\cos 2A \neq 2 \cos A.$$



FIG. 27.

4. Let ABC be a triangle, right-angled at C : make $AD=AC$ and $BD=BC$, produce DB and AC to meet in E .

Take $AC=1$ and $\angle BAC=A$,
and shew, when A is any acute
angle less than 45° , that

$$\tan 2A \neq 2 \tan A.$$

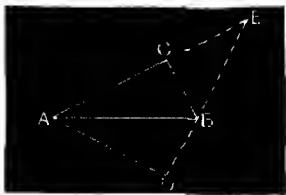


FIG. 28.

In Expts. 21, 22 the sines, cosines and tangents of given angles have been found: if the ratios are given the angles can be found by similar methods.

23. Find an angle whose sine is $\frac{m}{n}$.

Draw a quadrant of a circle with a convenient radius whose length represents n units on some chosen scale. Draw OX ,

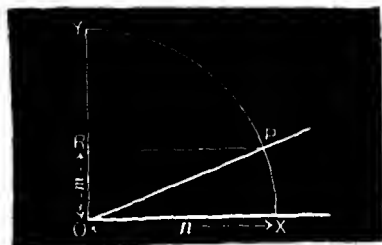


FIG. 29.

OY two radii at right angles. cut off OR along OY so that $OR = m$ units on the chosen scale. Draw $RP \parallel OX$ to cut the arc in P .

Then $\angle XOP$ is an angle whose sine is $\frac{m}{n}$. [Why?]

An alternative construction may be verified:

Take $OR = m$ units and draw RP perpendicular to OR . With centre O and radius $= n$ units describe an arc cutting R in P .

Then

$$\sin RPO = \frac{m}{n}.$$



FIG. 30.

24. Find an angle whose cosine is $\frac{m}{n}$.

25. Find an angle whose tangent is $\frac{m}{n}$.

The notation most commonly used for denoting angles whose sines, cosines or tangents are given quantities is the following:

$\sin^{-1}x \equiv$ an angle whose sine is x ,

$\cos^{-1}y \equiv$ an angle whose cosine is y ,

$\tan^{-1}z \equiv$ an angle whose tangent is z .

It must be carefully observed that although the “ -1 ” occupies the position of an index, yet $\sin^{-1}x$ must not be confused with $(\sin x)^{-1}$, i.e. $1 \div \sin x$. This is all the more important because in all other cases the powers of trigonometrical ratios are indicated by attaching the index to the name of the ratio.

Thus

$$\sin^2 A \equiv (\sin A)^2,$$

$$\tan^{\frac{1}{2}} 2A \equiv \sqrt{\tan 2A}.$$

This exceptional use of the symbol $^{-1}$, however, need not lead to any confusion; for the reciprocals of the sine, cosine and tangent have, as will be seen in § 8, special names given to them, so that the expressions

$$(\sin x)^{-1}, (\cos x)^{-1}, (\tan x)^{-1}$$

are not used.

Note. The continental notation for $\sin^{-1}x$, $\cos^{-1}x$, etc. is $\text{arc-sin } x$, $\text{arc-cos } x$, etc.

It should also be noticed that in the expression $\sin^{-1}x$, the symbol x stands for a number; in $\sin A$, the symbol A stands for an angle.

1. With scale and compasses make angles

(i) whose sines are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$;

(ii) whose cosines are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$;

(iii) whose tangents are $\frac{1}{2}$, 2 , $\frac{3}{2}$.

2. Find the number of degrees in each of the above angles by means of the tables constructed in Expts. 21, 22. Verify with a protractor or scale of chords.

§ 7. Advantages and use of tables.

When solving problems in "Heights and Distances" the degree of accuracy attainable by the method of drawing to scale, given in § 5, is very limited. Measurements to $\frac{1}{100}$ in. in the length of a line, and to $\frac{1}{2}^\circ$ in the magnitude of an angle, are only to be trusted when using extremely accurate instruments and with most careful drawing. Thus when finding the height of a hill of, say, 4000 ft. by the observed depressions of two consecutive milestones, suppose a length of about 5 in. is taken to represent a mile, according to which the height of the hill would be represented by less than 4 in.; then an error in drawing of .01 in. would give an error in the height of the hill

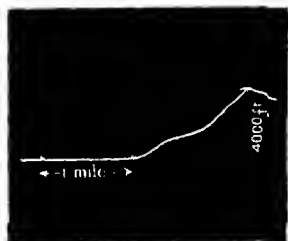


FIG. 31.

$$= \frac{.01}{4} \times 4000 \text{ ft.}$$

$$= 10 \text{ ft.}$$

Again, very rarely would the landmarks be so conveniently placed that the depressions could be expressed exactly in half-degrees. A much greater error will in general be introduced by inaccuracy in drawing these angles (Expt. 26) of depression or elevation as the case may be.

26. From the top of a hill the angles of depression of two landmarks known to be 120 yds. apart are 19° and 25° respectively. (Cf. Expt. 18.)

Using a large sheet of paper, take 3" to represent 120 yds., and find the several heights of the hill given by drawing diagrams

- (i) in which an error of half a degree is made in drawing the angle 19° (2 figures);
- (ii) in which an error of half a degree is made in drawing the angle 25° (2 figures);
- (iii) in which an error of half a degree is made in drawing both angles (4 figures).

27. Calculate the height of the hill, using the values of the tangents found in Expt. 22, as follows.

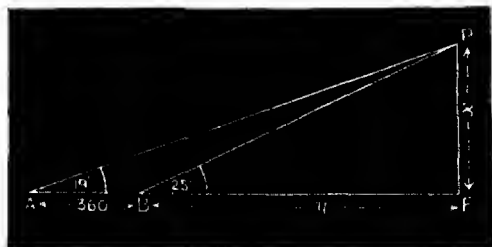


FIG. 32.

Let A and B be the landmarks 120 yds. apart, P the top of the hill, F the foot of the vertical through P; let $PF = x$ feet, $BF = y$ feet.

Then

$$AF = y + 360;$$

$$\therefore \frac{x}{y + 360} = \tan 19^\circ$$

$$\text{and } \frac{x}{y} = \tan 25^\circ;$$

$$\therefore \frac{y + 360}{x} - \frac{y}{x} = \frac{1}{\tan 19^\circ} - \frac{1}{\tan 25^\circ};$$

$$\begin{aligned}
 \therefore \frac{y+360}{x} - \frac{y}{x} &= \frac{360}{x} = \tan 71^\circ - \tan 65^\circ, \quad [\text{see p. 28}] \\
 &= 2.904 - 2.145, \\
 &= .759; \\
 \therefore x &= \frac{360}{.759} \\
 &= 474.3 \text{ feet.}
 \end{aligned}$$

The values used are practically extracts from a table of tangents correct to *three decimal places*. By methods given in more advanced Trigonometry, tables can be calculated to any required number of decimal places. Four- and five-figure tables are published for general use in Physics and by engineers; surveyors use seven-figure tables, and for some of the most exact astronomical calculations ten- and even twelve-figure tables are used.

28. Calculate the height of the hill in Expt. 27, using five-figure tables.

[We find from the tables on pp. 38, 39.

$$\tan 71^\circ = 2.90421,$$

$$\tan 65^\circ = 2.14451,$$

and hence $x = 473.87$ ft.]

This result is the true height to a hundredth of an inch, the result obtained by use of seven-figure tables being 473.8688 ft.

In general, four-, five- and seven-figure tables give results in problems such as the above which are correct to four, five and seven significant figures respectively.

| Sines | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|-------|--------|--------|--------|--------|--------|--------|--------|----------|
| 0° | ·00000 | ·00291 | ·00582 | ·00873 | ·01164 | ·01454 | ·01745 | 89° |
| 1 | ·01745 | ·02036 | ·02327 | ·02618 | ·02908 | ·03199 | ·03490 | 88 |
| 2 | ·03490 | ·03781 | ·04071 | ·04362 | ·04653 | ·04943 | ·05234 | 87 |
| 3 | ·05234 | ·05524 | ·05814 | ·06105 | ·06395 | ·06685 | ·06976 | 86 |
| 4 | ·06976 | ·07266 | ·07556 | ·07846 | ·08136 | ·08426 | ·08716 | 85 |
| 5 | ·08716 | ·09005 | ·09295 | ·09585 | ·09874 | ·10164 | ·10453 | 84 |
| 6 | ·10453 | ·10742 | ·11031 | ·11320 | ·11609 | ·11898 | ·12187 | 83 |
| 7 | ·12187 | ·12476 | ·12764 | ·13053 | ·13341 | ·13629 | ·13917 | 82 |
| 8 | ·13917 | ·14205 | ·14493 | ·14781 | ·15069 | ·15356 | ·15643 | 81 |
| 9 | ·15643 | ·15931 | ·16218 | ·16505 | ·16792 | ·17078 | ·17365 | 80 |
| 10 | ·17365 | ·17651 | ·17937 | ·18224 | ·18509 | ·18795 | ·19081 | 79 |
| 11 | ·19081 | ·19366 | ·19652 | ·19937 | ·20222 | ·20507 | ·20791 | 78 |
| 12 | ·20791 | ·21076 | ·21360 | ·21644 | ·21928 | ·22212 | ·22495 | 77 |
| 13 | ·22495 | ·22778 | ·23062 | ·23345 | ·23627 | ·23910 | ·24192 | 76 |
| 14 | ·24192 | ·24474 | ·24756 | ·25038 | ·25320 | ·25601 | ·25882 | 75 |
| 15 | ·25882 | ·26163 | ·26443 | ·26724 | ·27004 | ·27284 | ·27564 | 74 |
| 16 | ·27564 | ·27843 | ·28123 | ·28402 | ·28680 | ·28959 | ·29237 | 73 |
| 17 | ·29237 | ·29515 | ·29793 | ·30071 | ·30348 | ·30625 | ·30902 | 72 |
| 18 | ·30902 | ·31178 | ·31454 | ·31730 | ·32006 | ·32282 | ·32557 | 71 |
| 19 | ·32557 | ·32832 | ·33106 | ·33381 | ·33655 | ·33929 | ·34202 | 70 |
| 20 | ·34202 | ·34475 | ·34748 | ·35021 | ·35293 | ·35565 | ·35837 | 69 |
| 21 | ·35837 | ·36108 | ·36379 | ·36650 | ·36921 | ·37191 | ·37461 | 68 |
| 22 | ·37461 | ·37730 | ·37999 | ·38268 | ·38537 | ·38805 | ·39073 | 67 |
| 23 | ·39073 | ·39341 | ·39608 | ·39875 | ·40142 | ·40408 | ·40674 | 66 |
| 24 | ·40674 | ·40939 | ·41204 | ·41469 | ·41734 | ·41998 | ·42262 | 65 |
| 25 | ·42262 | ·42525 | ·42788 | ·43051 | ·43313 | ·43575 | ·43837 | 64 |
| 26 | ·43837 | ·44098 | ·44359 | ·44620 | ·44880 | ·45140 | ·45399 | 63 |
| 27 | ·45399 | ·45658 | ·45917 | ·46175 | ·46433 | ·46690 | ·46947 | 62 |
| 28 | ·46947 | ·47204 | ·47460 | ·47716 | ·47971 | ·48226 | ·48481 | 61 |
| 29 | ·48481 | ·48735 | ·48989 | ·49242 | ·49495 | ·49748 | ·50000 | 60 |
| 30 | ·50000 | ·50252 | ·50503 | ·50754 | ·51004 | ·51254 | ·51504 | 59 |
| 31 | ·51504 | ·51753 | ·52002 | ·52250 | ·52498 | ·52745 | ·52992 | 58 |
| 32 | ·52992 | ·53238 | ·53484 | ·53730 | ·53975 | ·54220 | ·54464 | 57 |
| 33 | ·54464 | ·54708 | ·54951 | ·55194 | ·55436 | ·55678 | ·55919 | 56 |
| 34 | ·55919 | ·56160 | ·56401 | ·56641 | ·56880 | ·57119 | ·57358 | 55 |
| 35 | ·57358 | ·57596 | ·57833 | ·58070 | ·58307 | ·58543 | ·58779 | 54 |
| 36 | ·58779 | ·59014 | ·59248 | ·59482 | ·59716 | ·59949 | ·60182 | 53 |
| 37 | ·60182 | ·60414 | ·60645 | ·60876 | ·61107 | ·61337 | ·61567 | 52 |
| 38 | ·61567 | ·61795 | ·62024 | ·62251 | ·62479 | ·62706 | ·62932 | 51 |
| 39 | ·62932 | ·63158 | ·63383 | ·63608 | ·63832 | ·64056 | ·64279 | 50 |
| 40 | ·64279 | ·64501 | ·64723 | ·64945 | ·65166 | ·65386 | ·65606 | 49 |
| 41 | ·65606 | ·65825 | ·66044 | ·66262 | ·66480 | ·66697 | ·66913 | 48 |
| 42 | ·66913 | ·67129 | ·67344 | ·67559 | ·67773 | ·67987 | ·68200 | 47 |
| 43 | ·68200 | ·68412 | ·68624 | ·68835 | ·69046 | ·69256 | ·69466 | 46 |
| 44 | ·69466 | ·69675 | ·69883 | ·70091 | ·70298 | ·70505 | ·70711 | 45 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Co-sines |

| Sines | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|-------|-------|-------|-------|-------|-------|-------|---------|---------|
| 45° | 70711 | 70916 | 71121 | 71325 | 71529 | 71732 | 71934 | 44° |
| 46 | 71934 | 72136 | 72337 | 72537 | 72737 | 72837 | 73135 | 43 |
| 47 | 73135 | 73333 | 73531 | 73728 | 73924 | 74120 | 74314 | 42 |
| 48 | 74314 | 74509 | 74703 | 74896 | 75088 | 75280 | 75471 | 41 |
| 49 | 75471 | 75661 | 75851 | 76041 | 76229 | 76417 | 76604 | 40 |
| 50 | 76604 | 76791 | 76977 | 77162 | 77347 | 77531 | 77715 | 39 |
| 51 | 77715 | 77897 | 78079 | 78261 | 78442 | 78622 | 78801 | 38 |
| 52 | 78801 | 78980 | 79158 | 79335 | 79512 | 79688 | 79864 | 37 |
| 53 | 79864 | 80038 | 80212 | 80386 | 80558 | 80730 | 80902 | 36 |
| 54 | 80902 | 81072 | 81242 | 81412 | 81580 | 81748 | 81915 | 35 |
| 55 | 81915 | 82082 | 82248 | 82413 | 82577 | 82741 | 82904 | 34 |
| 56 | 82904 | 83066 | 83228 | 83389 | 83549 | 83708 | 83867 | 33 |
| 57 | 83867 | 84025 | 84182 | 84339 | 84495 | 84650 | 84805 | 32 |
| 58 | 84805 | 84959 | 85112 | 85264 | 85416 | 85567 | 85717 | 31 |
| 59 | 85717 | 85866 | 86015 | 86163 | 86310 | 86457 | 86603 | 30 |
| 60 | 86603 | 86748 | 86892 | 87036 | 87178 | 87321 | 87462 | 29 |
| 61 | 87462 | 87603 | 87743 | 87882 | 88020 | 88158 | 88295 | 28 |
| 62 | 88295 | 88431 | 88566 | 88701 | 88835 | 88968 | 89101 | 27 |
| 63 | 89101 | 89232 | 89363 | 89493 | 89623 | 89751 | 89879 | 26 |
| 64 | 89879 | 90007 | 90133 | 90259 | 90383 | 90507 | 90631 | 25 |
| 65 | 90631 | 90753 | 90875 | 90996 | 91116 | 91236 | 91355 | 24 |
| 66 | 91355 | 91472 | 91590 | 91706 | 91822 | 91936 | 92050 | 23 |
| 67 | 92050 | 92164 | 92276 | 92388 | 92499 | 92609 | 92718 | 22 |
| 68 | 92718 | 92827 | 92935 | 93042 | 93148 | 93253 | 93358 | 21 |
| 69 | 93358 | 93462 | 93565 | 93667 | 93769 | 93869 | 93969 | 20 |
| 70 | 93969 | 94068 | 94167 | 94264 | 94361 | 94457 | 94552 | 19 |
| 71 | 94552 | 94646 | 94740 | 94832 | 94924 | 95015 | 95106 | 18 |
| 72 | 95106 | 95195 | 95284 | 95372 | 95459 | 95545 | 95630 | 17 |
| 73 | 95630 | 95715 | 95799 | 95882 | 95964 | 96046 | 96126 | 16 |
| 74 | 96126 | 96206 | 96285 | 96363 | 96440 | 96517 | 96593 | 15 |
| 75 | 96593 | 96667 | 96742 | 96815 | 96887 | 96959 | 97030 | 14 |
| 76 | 97030 | 97100 | 97169 | 97237 | 97304 | 97371 | 97437 | 13 |
| 77 | 97437 | 97502 | 97566 | 97630 | 97692 | 97754 | 97815 | 12 |
| 78 | 97815 | 97875 | 97934 | 97992 | 98050 | 98107 | 98163 | 11 |
| 79 | 98163 | 98218 | 98272 | 98325 | 98378 | 98430 | 98481 | 10 |
| 80 | 98481 | 98531 | 98580 | 98629 | 98676 | 98723 | 98769 | 9 |
| 81 | 98769 | 98814 | 98858 | 98902 | 98944 | 98986 | 99027 | 8 |
| 82 | 99027 | 99067 | 99106 | 99144 | 99182 | 99219 | 99255 | 7 |
| 83 | 99255 | 99290 | 99324 | 99357 | 99390 | 99421 | 99452 | 6 |
| 84 | 99452 | 99482 | 99511 | 99540 | 99567 | 99594 | 99619 | 5 |
| 85 | 99619 | 99644 | 99668 | 99692 | 99714 | 99736 | 99756 | 4 |
| 86 | 99756 | 99776 | 99795 | 99813 | 99831 | 99847 | 99863 | 3 |
| 87 | 99863 | 99878 | 99892 | 99905 | 99917 | 99929 | 99939 | 2 |
| 88 | 99939 | 99949 | 99958 | 99966 | 99973 | 99979 | 99985 | 1 |
| 89 | 99985 | 99989 | 99993 | 99996 | 99998 | 99999 | 1.00000 | 0 |
| | 80' | 50' | 40' | 30' | 20' | 10' | 0' | Cosines |

| Tan- gents | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|---------------|---------|---------|---------|---------|---------|---------|---------|-----------------|
| 0° | 0°00000 | 0°00291 | 0°00582 | 0°00873 | 0°01164 | 0°01455 | 0°01746 | 89° |
| 1 | 0°01746 | 0°02037 | 0°02328 | 0°02619 | 0°02910 | 0°03201 | 0°03492 | 88 |
| 2 | 0°03492 | 0°03783 | 0°04075 | 0°04366 | 0°04658 | 0°04949 | 0°05241 | 87 |
| 3 | 0°05241 | 0°05533 | 0°05824 | 0°06116 | 0°06408 | 0°06700 | 0°06993 | 86 |
| 4 | 0°06993 | 0°07285 | 0°07578 | 0°07870 | 0°08163 | 0°08456 | 0°08749 | 85 |
| 5 | 0°08749 | 0°09042 | 0°09335 | 0°09629 | 0°09923 | 0°10216 | 0°10510 | 84 |
| 6 | 0°10510 | 0°10805 | 0°11099 | 0°11394 | 0°11688 | 0°11983 | 0°12278 | 83 |
| 7 | 0°12278 | 0°12574 | 0°12869 | 0°13165 | 0°13461 | 0°13758 | 0°14054 | 82 |
| 8 | 0°14054 | 0°14351 | 0°14648 | 0°14945 | 0°15243 | 0°15540 | 0°15838 | 81 |
| 9 | 0°15838 | 0°16137 | 0°16435 | 0°16734 | 0°17033 | 0°17333 | 0°17633 | 80 |
| 10 | 0°17633 | 0°17933 | 0°18233 | 0°18534 | 0°18835 | 0°19136 | 0°19438 | 79 |
| 11 | 0°19438 | 0°19740 | 0°20042 | 0°20345 | 0°20648 | 0°20952 | 0°21256 | 78 |
| 12 | 0°21256 | 0°21560 | 0°21864 | 0°22169 | 0°22475 | 0°22781 | 0°23087 | 77 |
| 13 | 0°23087 | 0°23393 | 0°23700 | 0°24008 | 0°24316 | 0°24624 | 0°24933 | 76 |
| 14 | 0°24933 | 0°25242 | 0°25552 | 0°25862 | 0°26172 | 0°26483 | 0°26795 | 75 |
| 15 | 0°26795 | 0°27107 | 0°27419 | 0°27732 | 0°28046 | 0°28360 | 0°28675 | 74 |
| 16 | 0°28675 | 0°28990 | 0°29305 | 0°29621 | 0°29938 | 0°30255 | 0°30573 | 73 |
| 17 | 0°30573 | 0°30891 | 0°31210 | 0°31530 | 0°31850 | 0°32171 | 0°32492 | 72 |
| 18 | 0°32492 | 0°32814 | 0°33136 | 0°33460 | 0°33783 | 0°34108 | 0°34433 | 71 |
| 19 | 0°34433 | 0°34758 | 0°35085 | 0°35412 | 0°35740 | 0°36068 | 0°36397 | 70 |
| 20 | 0°36397 | 0°36727 | 0°37057 | 0°37388 | 0°37720 | 0°38053 | 0°38386 | 69 |
| 21 | 0°38386 | 0°38721 | 0°39055 | 0°39391 | 0°39727 | 0°40065 | 0°40403 | 68 |
| 22 | 0°40403 | 0°40741 | 0°41081 | 0°41421 | 0°41763 | 0°42105 | 0°42447 | 67 |
| 23 | 0°42447 | 0°42791 | 0°43136 | 0°43481 | 0°43828 | 0°44175 | 0°44523 | 66 |
| 24 | 0°44523 | 0°44872 | 0°45222 | 0°45573 | 0°45924 | 0°46277 | 0°46631 | 65 |
| 25 | 0°46631 | 0°46985 | 0°47341 | 0°47698 | 0°48055 | 0°48414 | 0°48773 | 64 |
| 26 | 0°48773 | 0°49134 | 0°49495 | 0°49858 | 0°50222 | 0°50587 | 0°50953 | 63 |
| 27 | 0°50953 | 0°51320 | 0°51688 | 0°52057 | 0°52427 | 0°52798 | 0°53171 | 62 |
| 28 | 0°53171 | 0°53545 | 0°53920 | 0°54296 | 0°54673 | 0°55051 | 0°55431 | 61 |
| 29 | 0°55431 | 0°55812 | 0°56194 | 0°56577 | 0°56962 | 0°57348 | 0°57735 | 60 |
| 30 | 0°57735 | 0°58124 | 0°58513 | 0°58905 | 0°59297 | 0°59691 | 0°60086 | 59 |
| 31 | 0°60086 | 0°60483 | 0°60881 | 0°61280 | 0°61681 | 0°62083 | 0°62487 | 58 |
| 32 | 0°62487 | 0°62892 | 0°63299 | 0°63707 | 0°64117 | 0°64528 | 0°64941 | 57 |
| 33 | 0°64941 | 0°65355 | 0°65771 | 0°66189 | 0°66608 | 0°67028 | 0°67451 | 56 |
| 34 | 0°67451 | 0°67875 | 0°68301 | 0°68728 | 0°69157 | 0°69588 | 0°70021 | 55 |
| 35 | 0°70021 | 0°70455 | 0°70891 | 0°71329 | 0°71769 | 0°72211 | 0°72654 | 54 |
| 36 | 0°72654 | 0°73100 | 0°73547 | 0°73996 | 0°74447 | 0°74900 | 0°75355 | 53 |
| 37 | 0°75355 | 0°75812 | 0°76272 | 0°76733 | 0°77196 | 0°77661 | 0°78129 | 52 |
| 38 | 0°78129 | 0°78598 | 0°79070 | 0°79544 | 0°80020 | 0°80498 | 0°80978 | 51 |
| 39 | 0°80978 | 0°81461 | 0°81946 | 0°82434 | 0°82923 | 0°83415 | 0°83910 | 50 |
| 40 | 0°83910 | 0°84407 | 0°84906 | 0°85408 | 0°85912 | 0°86419 | 0°86929 | 49 |
| 41 | 0°86929 | 0°87441 | 0°87955 | 0°88473 | 0°88992 | 0°89515 | 0°90040 | 48 |
| 42 | 0°90040 | 0°90569 | 0°91099 | 0°91633 | 0°92170 | 0°92709 | 0°93252 | 47 |
| 43 | 0°93252 | 0°93797 | 0°94345 | 0°94896 | 0°95451 | 0°96008 | 0°96569 | 46 |
| 44 | 0°96569 | 0°97133 | 0°97700 | 0°98270 | 0°98843 | 0°99420 | 1°00000 | 45 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cotan- gents |

| 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|----------|----------|----------|-----------|-----------|-----------|----------|-----------------|
| 1'00000 | 1'00583 | 1'01770 | 1'01761 | 1'02355 | 1'02952 | 1'03553 | 44° |
| 1'03553 | 1'04158 | 1'04766 | 1'05378 | 1'05994 | 1'06613 | 1'07237 | 43 |
| 1'07237 | 1'07864 | 1'08496 | 1'09131 | 1'09770 | 1'10414 | 1'11061 | 42 |
| 1'11061 | 1'11713 | 1'12369 | 1'13029 | 1'13694 | 1'14363 | 1'15037 | 41 |
| 1'15037 | 1'15715 | 1'16398 | 1'17085 | 1'17777 | 1'18474 | 1'19175 | 40 |
| 1'19175 | 1'19882 | 1'20593 | 1'21310 | 1'22031 | 1'22758 | 1'23490 | 39 |
| 1'23490 | 1'24227 | 1'24969 | 1'25717 | 1'26471 | 1'27230 | 1'27994 | 38 |
| 1'27994 | 1'28764 | 1'29541 | 1'30322 | 1'31110 | 1'31904 | 1'32704 | 37 |
| 1'32704 | 1'33511 | 1'34323 | 1'35142 | 1'35968 | 1'36800 | 1'37638 | 36 |
| 1'37638 | 1'38484 | 1'39336 | 1'40195 | 1'41061 | 1'41934 | 1'42815 | 35 |
| 1'42815 | 1'43703 | 1'44598 | 1'45501 | 1'46411 | 1'47330 | 1'48256 | 34 |
| 1'48256 | 1'49190 | 1'50133 | 1'51084 | 1'52043 | 1'53010 | 1'53987 | 33 |
| 1'53987 | 1'54972 | 1'55966 | 1'56969 | 1'57981 | 1'59002 | 1'60033 | 32 |
| 1'60033 | 1'61074 | 1'62125 | 1'63185 | 1'64256 | 1'65337 | 1'66428 | 31 |
| 1'66428 | 1'67530 | 1'68643 | 1'69766 | 1'70901 | 1'72047 | 1'73205 | 30 |
| 1'73205 | 1'74375 | 1'75556 | 1'76749 | 1'77955 | 1'79174 | 1'80405 | 29 |
| 1'80405 | 1'81649 | 1'82906 | 1'84177 | 1'85462 | 1'86760 | 1'88073 | 28 |
| 1'88073 | 1'89400 | 1'90741 | 1'92098 | 1'93470 | 1'94858 | 1'96261 | 27 |
| 1'96261 | 1'97681 | 1'99116 | 2'00569 | 2'02039 | 2'03526 | 2'05030 | 26 |
| 2'05030 | 2'06553 | 2'08094 | 2'09654 | 2'11233 | 2'12832 | 2'14451 | 25 |
| 2'14451 | 2'16090 | 2'17749 | 2'19430 | 2'21132 | 2'22857 | 2'24604 | 24 |
| 2'24604 | 2'26374 | 2'28167 | 2'29984 | 2'31826 | 2'33693 | 2'35585 | 23 |
| 2'35585 | 2'37504 | 2'39449 | 2'41421 | 2'43422 | 2'45451 | 2'47509 | 22 |
| 2'47509 | 2'49597 | 2'51715 | 2'53865 | 2'56046 | 2'58261 | 2'60529 | 21 |
| 2'60529 | 2'62791 | 2'65109 | 2'67462 | 2'69853 | 2'72281 | 2'74748 | 20 |
| 2'74748 | 2'77254 | 2'79802 | 2'82391 | 2'85023 | 2'87700 | 2'90421 | 19 |
| 2'90421 | 2'93189 | 2'96004 | 2'98869 | 3'01783 | 3'04749 | 3'07768 | 18 |
| 3'07768 | 3'10842 | 3'13972 | 3'17159 | 3'20406 | 3'23714 | 3'27085 | 17 |
| 3'27085 | 3'30521 | 3'34023 | 3'37594 | 3'41236 | 3'44951 | 3'48741 | 16 |
| 3'48741 | 3'52609 | 3'56557 | 3'60588 | 3'64705 | 3'68909 | 3'73205 | 15 |
| 3'73205 | 3'77595 | 3'82083 | 3'86671 | 3'91364 | 3'96165 | 4'01078 | 14 |
| 4'01078 | 4'06107 | 4'11256 | 4'16530 | 4'21933 | 4'27471 | 4'33148 | 13 |
| 4'33148 | 4'38969 | 4'44942 | 4'51071 | 4'57363 | 4'63825 | 4'70463 | 12 |
| 4'70463 | 4'77286 | 4'84300 | 4'91516 | 4'98940 | 5'06584 | 5'14455 | 11 |
| 5'14455 | 5'22566 | 5'30928 | 5'39552 | 5'48451 | 5'57638 | 5'67128 | 10 |
| 5'67128 | 5'76937 | 5'87080 | 5'97576 | 6'08444 | 6'19703 | 6'31375 | 9 |
| 6'31375 | 6'43484 | 6'56055 | 6'69116 | 6'82694 | 6'96823 | 7'11537 | 8 |
| 7'11537 | 7'26873 | 7'42871 | 7'59575 | 7'77035 | 7'95302 | 8'14435 | 7 |
| 8'14435 | 8'34496 | 8'55555 | 8'77689 | 9'00983 | 9'25530 | 9'51436 | 6 |
| 9'51436 | 9'78817 | 10'07803 | 10'38540 | 10'71191 | 11'05943 | 11'43005 | 5 |
| 11'43005 | 11'82617 | 12'25051 | 12'70620 | 13'19688 | 13'72674 | 14'30067 | 4 |
| 14'30067 | 14'92442 | 15'60478 | 16'34986 | 17'16934 | 18'07498 | 19'08114 | 3 |
| 19'08114 | 20'20555 | 21'47040 | 22'90377 | 24'54176 | 26'43160 | 28'63625 | 2 |
| 28'63625 | 31'24158 | 34'36777 | 38'18846 | 42'96408 | 49'10388 | 57'28996 | 1 |
| 57'28996 | 68'75009 | 85'93979 | 114'58865 | 171'88540 | 343'77371 | ∞ | 0 |
| 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cotan- gents |

| Sec- ants | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|--------------|---------|------------|---------|---------|---------|---------|---------|-------------|
| 0° | 1'00000 | 1'00000(4) | 1'00001 | 1'00004 | 1'00007 | 1'00011 | 1'00015 | 89° |
| 1 | 1'00015 | 1'00021 | 1'00027 | 1'00034 | 1'00042 | 1'00051 | 1'00061 | 88 |
| 2 | 1'00061 | 1'00072 | 1'00083 | 1'00095 | 1'00108 | 1'00122 | 1'00137 | 87 |
| 3 | 1'00137 | 1'00153 | 1'00169 | 1'00187 | 1'00205 | 1'00224 | 1'00244 | 86 |
| 4 | 1'00244 | 1'00265 | 1'00287 | 1'00309 | 1'00333 | 1'00357 | 1'00382 | 85 |
| 5 | 1'00382 | 1'00408 | 1'00435 | 1'00463 | 1'00491 | 1'00521 | 1'00551 | 84 |
| 6 | 1'00551 | 1'00582 | 1'00614 | 1'00647 | 1'00681 | 1'00715 | 1'00751 | 83 |
| 7 | 1'00751 | 1'00787 | 1'00825 | 1'00863 | 1'00902 | 1'00942 | 1'00983 | 82 |
| 8 | 1'00983 | 1'01024 | 1'01067 | 1'01111 | 1'01155 | 1'01200 | 1'01247 | 81 |
| 9 | 1'01247 | 1'01294 | 1'01342 | 1'01391 | 1'01440 | 1'01491 | 1'01543 | 80 |
| 10 | 1'01543 | 1'01595 | 1'01649 | 1'01703 | 1'01758 | 1'01815 | 1'01872 | 79 |
| 11 | 1'01872 | 1'01930 | 1'01989 | 1'02049 | 1'02110 | 1'02171 | 1'02234 | 78 |
| 12 | 1'02234 | 1'02298 | 1'02362 | 1'02428 | 1'02494 | 1'02562 | 1'02630 | 77 |
| 13 | 1'02630 | 1'02700 | 1'02770 | 1'02842 | 1'02914 | 1'02987 | 1'03061 | 76 |
| 14 | 1'03061 | 1'03137 | 1'03213 | 1'03290 | 1'03368 | 1'03447 | 1'03528 | 75 |
| 15 | 1'03528 | 1'03609 | 1'03691 | 1'03774 | 1'03858 | 1'03944 | 1'04030 | 74 |
| 16 | 1'04030 | 1'04117 | 1'04206 | 1'04295 | 1'04385 | 1'04477 | 1'04569 | 73 |
| 17 | 1'04569 | 1'04663 | 1'04757 | 1'04853 | 1'04950 | 1'05047 | 1'05146 | 72 |
| 18 | 1'05146 | 1'05246 | 1'05347 | 1'05449 | 1'05552 | 1'05657 | 1'05762 | 71 |
| 19 | 1'05762 | 1'05869 | 1'05976 | 1'06085 | 1'06195 | 1'06306 | 1'06418 | 70 |
| 20 | 1'06418 | 1'06531 | 1'06645 | 1'06761 | 1'06878 | 1'06995 | 1'07115 | 69 |
| 21 | 1'07115 | 1'07235 | 1'07356 | 1'07479 | 1'07602 | 1'07727 | 1'07853 | 68 |
| 22 | 1'07853 | 1'07981 | 1'08109 | 1'08239 | 1'08370 | 1'08503 | 1'08636 | 67 |
| 23 | 1'08636 | 1'08771 | 1'08907 | 1'09044 | 1'09183 | 1'09323 | 1'09464 | 66 |
| 24 | 1'09464 | 1'09606 | 1'09750 | 1'09895 | 1'10041 | 1'10189 | 1'10338 | 65 |
| 25 | 1'10338 | 1'10488 | 1'10640 | 1'10793 | 1'10947 | 1'11103 | 1'11260 | 64 |
| 26 | 1'11260 | 1'11419 | 1'11579 | 1'11740 | 1'11903 | 1'12067 | 1'12233 | 63 |
| 27 | 1'12233 | 1'12400 | 1'12568 | 1'12738 | 1'12910 | 1'13083 | 1'13257 | 62 |
| 28 | 1'13257 | 1'13433 | 1'13610 | 1'13789 | 1'13970 | 1'14152 | 1'14335 | 61 |
| 29 | 1'14335 | 1'14521 | 1'14707 | 1'14896 | 1'15085 | 1'15277 | 1'15470 | 60 |
| 30 | 1'15470 | 1'15665 | 1'15861 | 1'16059 | 1'16259 | 1'16460 | 1'16663 | 59 |
| 31 | 1'16663 | 1'16868 | 1'17075 | 1'17283 | 1'17493 | 1'17704 | 1'17918 | 58 |
| 32 | 1'17918 | 1'18133 | 1'18350 | 1'18569 | 1'18790 | 1'19012 | 1'19236 | 57 |
| 33 | 1'19236 | 1'19463 | 1'19691 | 1'19920 | 1'20152 | 1'20386 | 1'20622 | 56 |
| 34 | 1'20622 | 1'20859 | 1'21099 | 1'21341 | 1'21584 | 1'21830 | 1'22077 | 55 |
| 35 | 1'22077 | 1'22327 | 1'22579 | 1'22833 | 1'23089 | 1'23347 | 1'23607 | 54 |
| 36 | 1'23607 | 1'23869 | 1'24136 | 1'24400 | 1'24669 | 1'24940 | 1'25214 | 53 |
| 37 | 1'25214 | 1'25489 | 1'25767 | 1'26047 | 1'26330 | 1'26615 | 1'26902 | 52 |
| 38 | 1'26902 | 1'27191 | 1'27483 | 1'27778 | 1'28075 | 1'28374 | 1'28676 | 51 |
| 39 | 1'28676 | 1'28980 | 1'29287 | 1'29597 | 1'29909 | 1'30223 | 1'30541 | 50 |
| 40 | 1'30541 | 1'30861 | 1'31183 | 1'31509 | 1'31837 | 1'32168 | 1'32501 | 49 |
| 41 | 1'32501 | 1'32838 | 1'33177 | 1'33519 | 1'33864 | 1'34212 | 1'34563 | 48 |
| 42 | 1'34563 | 1'34917 | 1'35274 | 1'35634 | 1'35997 | 1'36363 | 1'36733 | 47 |
| 43 | 1'36733 | 1'37105 | 1'37481 | 1'37860 | 1'38242 | 1'38628 | 1'39016 | 46 |
| 44 | 1'39016 | 1'39409 | 1'39804 | 1'40203 | 1'40606 | 1'41012 | 1'41421 | 45 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Co- sine |

| | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|----|----------|----------|----------|-----------|-----------|-----------|----------|-----------|
| 1 | 1'41421 | 1'41835 | 1'42251 | 1'42672 | 1'43096 | 1'43524 | 1'43956 | 44° |
| 2 | 1'43956 | 1'44391 | 1'44831 | 1'45274 | 1'45721 | 1'46173 | 1'46628 | 43 |
| 3 | 1'46628 | 1'47087 | 1'47551 | 1'48019 | 1'48491 | 1'48967 | 1'49448 | 42 |
| 4 | 1'49448 | 1'49933 | 1'50422 | 1'50916 | 1'51415 | 1'51918 | 1'52425 | 41 |
| 5 | 1'52425 | 1'52938 | 1'53455 | 1'53977 | 1'54504 | 1'55036 | 1'55572 | 40 |
| 6 | 1'55572 | 1'56114 | 1'56661 | 1'57213 | 1'57771 | 1'58333 | 1'58902 | 39 |
| 7 | 1'58902 | 1'59475 | 1'60054 | 1'60639 | 1'61229 | 1'61825 | 1'62427 | 38 |
| 8 | 1'62427 | 1'63035 | 1'63648 | 1'64268 | 1'64894 | 1'65526 | 1'66164 | 37 |
| 9 | 1'66164 | 1'66809 | 1'67460 | 1'68117 | 1'68782 | 1'69452 | 1'70130 | 36 |
| 10 | 1'70130 | 1'70815 | 1'71506 | 1'72205 | 1'72911 | 1'73624 | 1'74345 | 35 |
| 11 | 1'74345 | 1'75073 | 1'75808 | 1'76552 | 1'77303 | 1'78062 | 1'78829 | 34 |
| 12 | 1'78829 | 1'79604 | 1'80388 | 1'81180 | 1'81981 | 1'82790 | 1'83608 | 33 |
| 13 | 1'83608 | 1'84435 | 1'85271 | 1'86116 | 1'86990 | 1'87834 | 1'88708 | 32 |
| 14 | 1'88708 | 1'89591 | 1'90485 | 1'91388 | 1'92302 | 1'93226 | 1'94160 | 31 |
| 15 | 1'94160 | 1'95106 | 1'96062 | 1'97029 | 1'98008 | 1'98998 | 2'00000 | 30 |
| 16 | 2'00000 | 2'01014 | 2'02039 | 2'03077 | 2'04128 | 2'05191 | 2'06267 | 29 |
| 17 | 2'06267 | 2'07356 | 2'08458 | 2'09574 | 2'10704 | 2'11847 | 2'13005 | 28 |
| 18 | 2'13005 | 2'14178 | 2'15366 | 2'16568 | 2'17786 | 2'19019 | 2'20269 | 27 |
| 19 | 2'20269 | 2'21535 | 2'22817 | 2'24116 | 2'25432 | 2'26766 | 2'28117 | 26 |
| 20 | 2'28117 | 2'29487 | 2'30875 | 2'32282 | 2'33708 | 2'35154 | 2'36620 | 25 |
| 21 | 2'36620 | 2'38107 | 2'39614 | 2'41142 | 2'42692 | 2'44264 | 2'45859 | 24 |
| 22 | 2'45859 | 2'47477 | 2'49119 | 2'50784 | 2'52474 | 2'54190 | 2'55930 | 23 |
| 23 | 2'55930 | 2'57698 | 2'59491 | 2'61313 | 2'63162 | 2'65040 | 2'66947 | 22 |
| 24 | 2'66947 | 2'68884 | 2'70851 | 2'72850 | 2'74881 | 2'76945 | 2'79043 | 21 |
| 25 | 2'79043 | 2'81175 | 2'83342 | 2'85545 | 2'87785 | 2'90063 | 2'92380 | 20 |
| 26 | 2'92380 | 2'94737 | 2'97135 | 2'99574 | 3'02057 | 3'04584 | 3'07155 | 19 |
| 27 | 3'07155 | 3'09774 | 3'12440 | 3'15155 | 3'17920 | 3'20737 | 3'23607 | 18 |
| 28 | 3'23607 | 3'26531 | 3'29512 | 3'32551 | 3'35649 | 3'38808 | 3'42030 | 17 |
| 29 | 3'42030 | 3'45317 | 3'48071 | 3'52094 | 3'55587 | 3'59154 | 3'62796 | 16 |
| 30 | 3'62796 | 3'66515 | 3'70315 | 3'74198 | 3'78166 | 3'82223 | 3'86370 | 15 |
| 31 | 3'86370 | 3'90613 | 3'94952 | 3'99393 | 4'03938 | 4'08591 | 4'13357 | 14 |
| 32 | 4'13357 | 4'18238 | 4'23239 | 4'28366 | 4'33622 | 4'39012 | 4'44541 | 13 |
| 33 | 4'44541 | 4'50216 | 4'56041 | 4'62023 | 4'68167 | 4'74482 | 4'80973 | 12 |
| 34 | 4'80973 | 4'87649 | 4'94517 | 5'01585 | 5'08863 | 5'16359 | 5'24084 | 11 |
| 35 | 5'24084 | 5'32049 | 5'40263 | 5'48740 | 5'57493 | 5'66533 | 5'75877 | 10 |
| 36 | 5'75877 | 5'85539 | 5'95536 | 6'05886 | 6'16607 | 6'27719 | 6'39245 | 9 |
| 37 | 6'39245 | 6'51208 | 6'63633 | 6'76547 | 6'89979 | 7'03962 | 7'18530 | 8 |
| 38 | 7'18530 | 7'33719 | 7'49571 | 7'66130 | 7'83443 | 8'01565 | 8'20551 | 7 |
| 39 | 8'20551 | 8'40466 | 8'61379 | 8'83367 | 9'06515 | 9'30917 | 9'56677 | 6 |
| 40 | 9'56677 | 9'83912 | 10'12752 | 10'43343 | 10'75849 | 11'10455 | 11'47371 | 5 |
| 41 | 11'47371 | 11'86837 | 12'29125 | 12'74550 | 13'23472 | 13'76312 | 14'33559 | 4 |
| 42 | 14'33559 | 14'95788 | 15'63739 | 16'38041 | 17'19843 | 18'10262 | 19'10732 | 3 |
| 43 | 19'10732 | 20'23028 | 21'49367 | 22'92559 | 24'56212 | 26'45051 | 28'65371 | 2 |
| 44 | 28'65371 | 31'25758 | 34'38232 | 38'20155 | 42'97571 | 49'11406 | 57'29869 | 1 |
| 45 | 57'29869 | 68'75736 | 85'94561 | 114'59301 | 171'88831 | 343'77516 | ∞ | 0 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cosecants |

EXERCISES.

[The following exercises are to be worked with the tables on pp. 36—41.

The answers should be roughly verified by a figure drawn to scale.]

1. Two adjacent sides of a parallelogram are of length 15 and 24 units respectively, and the angle between them is 60° ; find the lengths of both diagonals.

2. A rectangular garden, 30 yds. long and 25 yds. wide, is watered by a hose, which will deliver water to a distance of 22 feet from the nozzle. What length of hose is required to reach every part of the garden, if it is attached to a stand-pipe in the middle of one of the longer sides of the garden?

3. A dog-kennel is to have its roof covered with felt. The kennel is 6 feet long, 4 feet broad and its height is 5 feet to the ridge and 3 feet 9 inches to the eaves. How many square feet of felt are required?

4. A wooden foot-bridge is thrown across a railway cutting, 30 feet in depth; the width of the railroad is 27 feet and the banks have a slope of 35° : find the length of the bridge, allowing 6 feet overlap at each end.

5. A person, whose eyes are 5 ft. 6 in. above the ground, is standing within a railway tunnel whose height is 25 feet. The elevations of the tops of the arches at its ends are $5^\circ 37'$, $13^\circ 22'$ respectively. Find the length of the tunnel.

6. Two persons, at stations 1000 yds. apart, due East and West, simultaneously observe a balloon. At the westerly station the balloon bears N.E. by E. and its elevation is 47° ; at the other station it bears N.W. by N.: shew that its elevation is $58^\circ 4'$ at this station, and find the height of the balloon.

7. A church stands in the centre of a square, the summit of the spire being vertically over the middle point of the square. When the sun has an altitude of $33^\circ 24'$, the shadow of the spire just reaches a corner of the square. If the square contains $2\frac{1}{2}$ acres find the height of the spire.

8. A ring 10 inches in diameter is suspended by six equal strings, attached to its circumference at equal intervals, from a point 1 foot above its centre: find the angle between two consecutive strings.

9. The guns from a fort on the top of a cliff will carry a distance of $1\frac{1}{2}$ miles; they can be depressed 15° for point-blank range: what is the breadth of the danger-zone, if the height of the cliff is 250 feet?

10. A gun, pointing through an embrasure of a fort can swing 15° to either side. A ship steaming at 22 knots an hour passes the fort at a least distance of one knot: how long is she under fire from the gun?

11. A man, on the top of a hill, observes the angles of depression of the top and bottom of a tower 50 feet high at the foot of the hill to be $5^\circ 49'$, $6^\circ 2''$ respectively: find the height of the hill.

12. A tower stands on the side of a hill, rising 1 in 25 uniformly. After proceeding 200 feet straight up the hill the angle of elevation of the top of the tower is $34^\circ 22'$. Find the height of the tower.

§ 8. Connections between the trigonometrical ratios of a given angle.

Besides the sine, cosine and tangent of an angle other trigonometrical ratios are used.

Those in general use are the reciprocals of the sine, cosine and tangent. These are called respectively the **cosecant**, **secant** and **cotangent**; and the abbreviations cosec., sec., tan. are used.

Thus we have, in any right-angled triangle,

$$\begin{aligned}\text{cosec}(\text{angle}) &= \frac{1}{\sin(\text{angle})} = \frac{\text{side opposite right angle}}{\text{side opposite angle}}, \\ \sec(\text{angle}) &= \frac{1}{\cos(\text{angle})} = \frac{\text{side opposite right angle}}{\text{side opposite other acute angle}}, \\ \cot(\text{angle}) &= \frac{1}{\tan(\text{angle})} = \frac{\text{side opposite other acute angle}}{\text{side opposite angle}}.\end{aligned}$$

Two others are used, chiefly in arch construction, the versed-sine and the co-versed-sine. They are defined as follows:

$$\begin{aligned}\text{vers}(\text{angle}) &= 1 - \cos(\text{angle}), \\ \text{covers}(\text{angle}) &= 1 - \sin(\text{angle}).\end{aligned}$$

DERIVATIONS OF THE NAMES OF THE RATIOS 45

The derivation of the names of the ratios is instructive. The earlier mathematicians defined the sine, cosine, etc. as the measures of certain lines in a circle whose radius was taken as the unit of length.

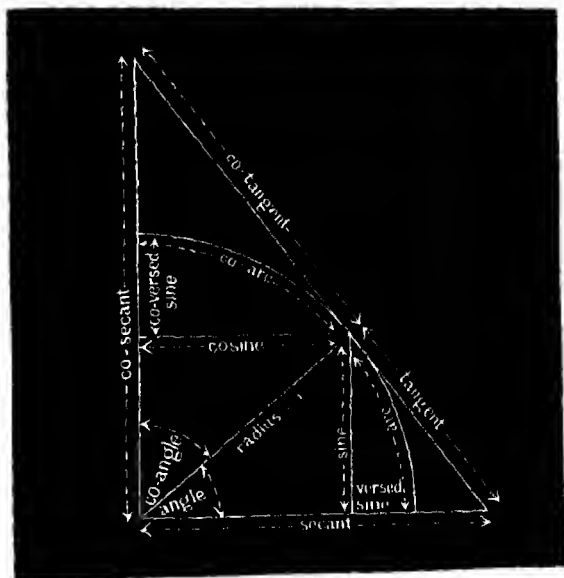


FIG. 33.

Fig. 33 shews these lines: and from it the derivation of "tangent" and "secant" are evident, whilst the word "arc" is the key to the derivation of the word "sine." The curve is looked on as the half of a bow (*L. arcus*) of which the "sine" is half the string by which the bow is bent (*L. sinus*, a bending; cf. *sinew*, *A.S. sinu*). The line denoting the versed sine was sometimes called the *sagitta* (*L. an arrow*). Originally the length of the whole chord of the double arc was given instead of the sine of the arc. The "versed sine" is the sine turned through a right angle about its foot, whilst its length decreases to allow its other extremity to remain on the circle.

The other four ratios are the ratios of the complementary angle, or arc as the early mathematicians considered it (see *Circular Measure*, § 20).

This notation was found inconvenient as the "unit" used for the radius had always to be stated, and the idea of ratios was introduced into general use by Euler about 1780.

29. Let BAC be any acute angle, P a point in one arm of the angle and PM perpendicular to the other arm.



FIG. 34.

Then by Pythagoras' Theorem, in the right-angled triangle AMP , we have

$$AM^2 + MP^2 = AP^2.$$

Dividing each side of this equation by (i) AP^2 , (ii) AM^2 , (iii) MP^2 , prove

$$(i) \quad \cos^2 A + \sin^2 A = 1,$$

$$(ii) \quad \sec^2 A - \tan^2 A = 1,$$

$$(iii) \quad \operatorname{cosec}^2 A - \cot^2 A = 1.$$

These connections between the trigonometrical ratios, together with those derived from the definitions already given, viz.:

$$(iv) \quad \tan A = \frac{\sin A}{\cos A},$$

$$(v) \quad \tan A \cdot \cot A = 1,$$

$$(vi) \quad \cos A \cdot \sec A = 1,$$

$$(vii) \quad \sin A \cdot \operatorname{cosec} A = 1,$$

enable us to find all the ratios of a given angle if one of them is given.

The values of all the ratios can be expressed in terms of any one ratio. Thus for $\tan A = t$, construct a triangle ABC , right-angled at B , in which $AB = 1$, $BC = t$; then,

VALUES OF RATIOS IN TERMS OF ANY ONE RATIO 47

since $AC^2 = AB^2 + BC^2$, the values of the other ratios can be at once written down.

$$\text{For, } \therefore AC = \sqrt{1+t^2}.$$

$$\therefore \sin \theta = \frac{CB}{CA} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan \theta}{\sqrt{1+\tan^2 \theta}}$$

$$\cos \theta = \frac{AB}{AC} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2 \theta}}$$

and so on.

The same results may be obtained by the use of the seven formulae on page 46.

For instance,

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}}.$$

The student should obtain, in this manner, the values of all the ratios in terms of any one ratio and tabulate them. [*See Answers.*]

In any particular example, however, it is often better to proceed from first principles, rather than to quote these relations.

30. Given $\sin P = \frac{40}{61}$, find $\tan A$, $\operatorname{cosec} A$.

Draw a triangle ABC right-angled at C . Let $AC=60$ units, $AB=61$ units (the triangle need not be drawn to scale).

Then $\angle ABC = P$. Shew that $BC=11$ units, and hence

$$\tan A = \frac{40}{60},$$

$$\operatorname{cosec} A = \frac{61}{40}.$$



FIG. 35.

EXERCISES.

1. Given that $\sin A = \frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.
2. Given that $\cos B = \frac{1}{3}$, find $\sin B$ and $\cot B$.
3. Given that $\tan A = \frac{4}{3}$, find $\sin A$ and $\sec A$.
4. Given that $\sec \theta = 4$, find $\cot \theta$ and $\sin \theta$.
5. Given that $\tan \theta = \sqrt{3}$, find $\sin \theta$ and $\cos \theta$.
6. Given that $\cot \theta = \frac{2}{\sqrt{5}}$, find $\sin \theta$ and $\sec \theta$.
7. Given that $\sin \theta = \frac{b}{c}$, find $\tan \theta$.
8. Given that $\tan \theta = \frac{a}{b}$, find $\sin \theta$ and $\cos \theta$.
9. Given that $\cos \theta = \frac{1}{a}$, find $\sin \theta$ and $\cot \theta$.
10. If $\sin \theta = a$, and $\tan \theta = b$, prove that $(1 - a^2)(1 + b^2) = 1$.
11. If $\cos \theta = h$, and $\tan \theta = k$, find the equation connecting h and k .

§9. Graphical representation of the trigonometrical ratios.

The tables given on pp. 36-41 for the trigonometrical ratios are too bulky to give a clear mental impression of the variation of the ratios for the whole range between 0° and 90° . If, however, rough approximations to the values there given are used to draw graphs, the diagrams obtained make the variation for the whole range evident at a glance.

31. Draw a graph of $y = \sin x$. Take two straight lines, Ox , Oy , at right angles to one another.

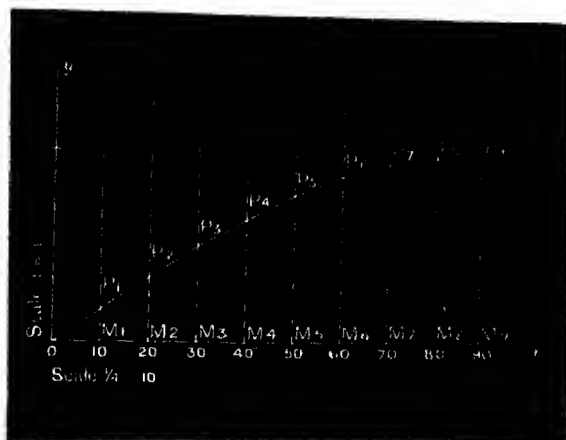


FIG. 36.

Take any convenient unit (say $\frac{1}{2}$ in.) along Ox to represent 10° , and mark off points $M_1, M_2, \dots M_9$, so that $OM_1, OM_2, \dots OM_9$ represent $10^\circ, 20^\circ, \dots 90^\circ$.

Take 2 in. along Oy to represent unity: then, using either a diagonal scale reading to $\cdot 01$ in. or squared paper ruled in inches and tenths, the values of the *sines* can be graphically shewn, *true to two places of decimals*, by erecting perpendiculars $M_1P_1, M_2P_2, \dots M_9P_9$ to represent on this scale (2 in. = 1) the values of $\sin 10^\circ, \sin 20^\circ, \dots \sin 90^\circ$, according to the following table, which is taken, true to two places, from the tables on pp. 36, 37. Fig. 36 is drawn half-size.

| | | | | | | | | | | |
|---------------|-----------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| x (angle) = | 0° | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90° |
| y (sine) = | 0.00 | 0.17 | 0.34 | 0.50 | 0.64 | 0.77 | 0.87 | 0.94 | 0.98 | 1.00 |

Join $OP_1, P_1P_2, \dots P_8P_9$. Observe that the angles which $OP_1, P_1P_2, \dots P_8P_9$ make with Ox are all different, and steadily decrease.

If, instead of the broken line $OP_1 \dots P_9$, a curve is drawn freehand, passing with a fair sweep through the points O, P_1, P_2, \dots, P_9 , a graph for $y = \sin x$ is obtained for *all* values of x between 0° and 90° . Observe that in no place does the curve depart very far from the broken line: thus the angle of slope of each of the straight lines $OP_1, P_1P_2, \dots, P_8P_9$ gives very nearly the average angle of slope of the curve for each interval of 10° . Also notice that the agreement between the parts of the curve and the straight lines becomes closer as the angle increases.

If the freehand curve is well drawn on a fairly large scale—the use of “French curves” or “Brooks’ flexible curves” are recommended—the value of $\sin x$ can be approximately determined for any acute angle, or the value of any acute angle x whose sine is given. The process is called **graphical interpolation**.

32. Find $\sin 39^\circ$ graphically.

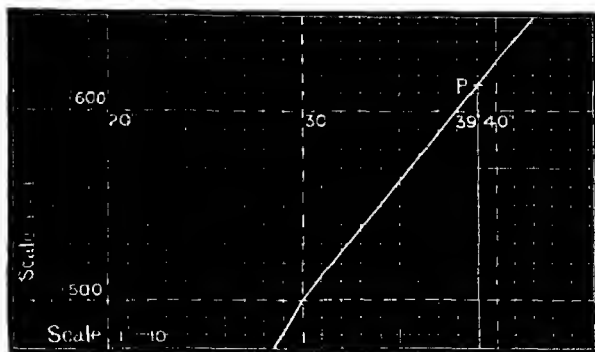


FIG. 37.

Construct a graph of $y = \sin x$ on a large sheet of paper, taking 1 in. along Ox to represent 10° , and 10 in. along or parallel to Oy to represent unity. A part of the graph is shewn in Fig. 37.

Take the point **M** so that **OM** represents 39° . erect a perpendicular **MP** cutting the graph of $y = \sin x$ in **P**, then **MP** represents (on the scale 10 in = 1) the value of $\sin 39^\circ$.

Measure **MP** with a diagonal scale, and compare the value obtained with the value given in the tables on p. 36

1 Find graphically the values of

$$(i) \sin 25^\circ, \sin 84^\circ, \sin 15^\circ, \sin 22\frac{1}{2}^\circ,$$

$$(ii) \sin^{-1} 0.309, \sin^{-1} 0.809, \sin^{-1} 0.951,$$

and verify the results by comparison with the tables on pp. 36, 37

33 Draw the graph of $y = \cos x$, for all values of x between 0° and 90° , taking from the tables on pp. 36, 37, the values of $\cos x$ for $x = 10^\circ, 20^\circ, \dots, 90^\circ$, and using the same scales as in Expt. 32. Find by graphical interpolation the values of $\cos 26^\circ, \cos 83^\circ$, and the angles $\cos^{-1} 0.970, \cos^{-1} 0.470$. Verify the results by comparison with the tables

34 Draw graphs of $y = \tan x, y = \cot x, y = \sec x, y = \operatorname{cosec} x$ for all values of x between 0° and 90°

2. Verify experimentally for the angles $21^\circ, 47^\circ, 62^\circ$, that

$$1 + \tan^2 x = \sec^2 x,$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x,$$

obtaining the values from the graphs drawn in Expt. 34.

3 Compare by superposition the general shapes of the graphs of $\sin x$ and $\cos x$, and deduce from them that

$$\sin x = \cos (90^\circ - x)$$

4. Shew, similarly, that

$$(1) \tan x = \cot (90^\circ - x),$$

$$(2) \sec x = \operatorname{cosec} (90^\circ - x)$$

§ 10. "Rate of Increase."

35. Using the same horizontal and vertical scales as in Expt. 32, mark a series of points whose heights above Ox represent the values of $y = \sin x$, true to three places of decimals, for intervals of 1° . Join consecutive points, and observe that this broken line is indistinguishable with the scale used (i.e. error indistinguishable = $\cdot 01$ in., representing $\cdot 001$ in the value of the sine) from the continuous curve forming the graph of $y = \sin x$. Also join the points corresponding to 10° , 20° , ... 90° , and shew that now the error in the height above Ox may be as much as $\cdot 06$ in., i.e. an error in the sine of $\cdot 006$.

36. Take 5 in. along Ox to represent 1° , and 2 in. along Oy to represent $\cdot 01$, and mark a series of points between (i) 5° and 6° , (ii) 41° and 42° , (iii) 78° and 79° , whose heights above Ox represent the values of $y = \sin x$, true to four places of decimals, for intervals of $1'$. Fig. 38 on the opposite page is for the interval 41° and 42° ; the straight lines Ox and Oy are 131 in. below AB , and 205 in. to the left of AC , respectively. Join consecutive points and shew that the broken line is indistinguishable from the continuous curve forming the graph of $y = \sin x$, and that the error in the value of the sine is less than $\cdot 0001$.

37. Repeat Expts. 35, 36 for the graphs of $y = \tan x$, and $y = \sec x$.

From the above experiments it is seen that, provided the points are taken close enough together, the graphs of trigonometrical ratios can be considered to coincide with the series of short straight lines joining consecutive points; and that the closer these points are taken together, the less is the error introduced by this assumption. This law will be found to be, with certain restrictions, a general law for all graphs.

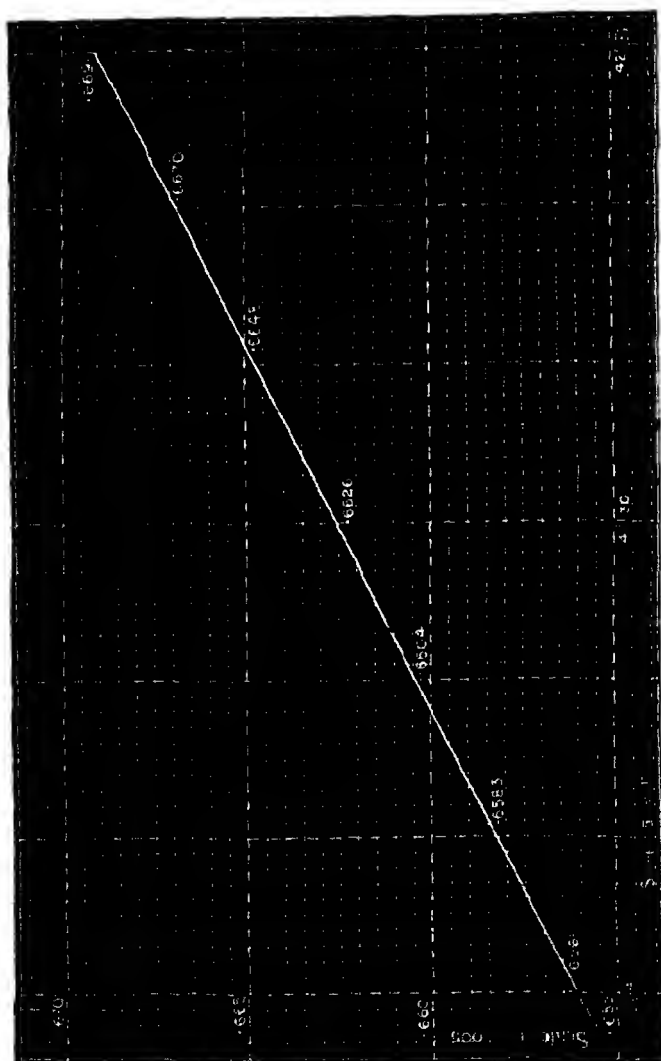


FIG. 38.

38. Draw two straight lines Ox, Oy at right angles to one another, and $APQR$ a straight line inclined to them.

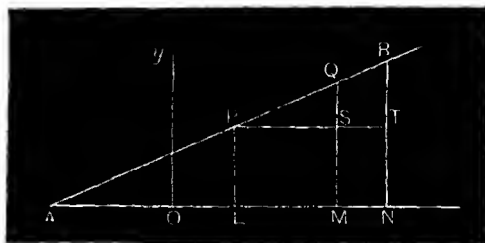


FIG 39

Draw PL, QM, RN perpendicular to Ox , and PST parallel to Ox , cutting QM, RN in S, T respectively. Then the angle RAN is the **angle of slope** of the line $APQR$: call this angle θ .

Then

$$\angle RPT = \theta.$$

$$\therefore \tan \theta = \frac{SQ}{PS} = \frac{MQ - MS}{LM} = \frac{MQ - LP}{OM - OL}.$$

$$\text{Similarly } \tan \theta = \frac{NR - LP}{ON - OL}.$$

$$\therefore \frac{MQ - LP}{OM - OL} = \frac{NR - LP}{ON - OL}.$$

The results obtained in Expt. 38 are most important; they may be expressed in words as follows:

I. The ratio of the difference of the ordinates (y 's) to the difference of the abscissae (x 's) of two points is equal to the tangent of the angle of slope of the straight line joining the two points.

II. "The rate of increase" of the ordinate along a straight line is constant.

§ 11. Tabular Interpolation.

39. Draw two straight lines Ox, Oy at right angles to one another. Take any two points P, R and draw PL, RN per-

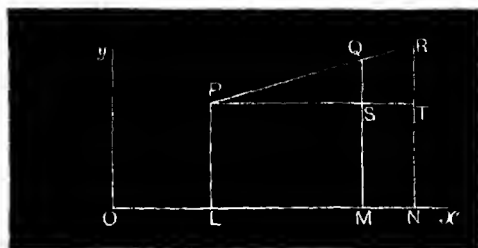


FIG. 40

pendicular to Ox and PT perpendicular to RN . Let OL, ON, LP, NR be x_1, x_2, y_1, y_2 respectively.

Take a point M on Ox , such that $OM = x$; draw MQ perpendicular to Ox , meeting PT in S and PR in Q . Let $MQ = y$

Then, by Expt. 38,

$$\frac{y_2 - y_1}{x_2 - x_1} = \tan RPT = m \text{ say,}$$

$$\frac{y - y_1}{x - x_1} = \tan RPT = m$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m,$$

$$\therefore y = y_1 + m(x - x_1)$$

This may be written

$$y = y_1 + \frac{x - x_1}{x_2 - x_1} (y_2 - y_1) \quad \dots \dots (1)$$

Similarly if x_1, y_1, x_2, y_2 , and y are given,

$$x = x_1 + m'(y - y_1) \text{ where } m' = \frac{x_2 - x_1}{y_2 - y_1},$$

$$\text{or } x = x_1 + \frac{y - y_1}{y_2 - y_1} (x_2 - x_1) \quad \dots \dots (2).$$

It follows that, if the values of x and y for two points on a straight line are known and the value of either x or y for any other point on the straight line is given, then the other value for the point can be found by either one or other of the formulae of Expt. 39.

Also, since it was deduced from Expts. 35, 36, 37 that any graph could be considered to be composed of a large number of very short straight lines joining near points on it, these formulae can be used to obtain the values of the x or y of any point on the graph for which the values are not tabulated, if one of them is given. This process is however only used, as a rule, for points *intermediate* between the given points.

These deductions may be enunciated as a law, generally called the "Rule of proportional differences" or the "Principle of proportional parts."

RULE. In any graph, the differences between the ordinates of any three points on it are proportional to the corresponding differences between the abscissae, provided these differences are small compared respectively with the ordinates or abscissae themselves.

With the meaning, given above, assigned to the words "provided the points are close enough together," the Rule may be applied to numerical or tabular interpolation with a set of tables without drawing the graph.

40. Find $\sin 35^\circ 28' 30''$.

It is found from the tables that

$$\sin 35^\circ 20' = 0.57833,$$

$$\sin 35^\circ 30' = 0.58070;$$

$$\therefore \sin 35^\circ 30' - \sin 35^\circ 20' = 0.00237.$$

But, by the "Rule of Proportional Differences,"

$$\frac{\sin 35^\circ 28' 30'' - \sin 35^\circ 20'}{\sin 35^\circ 30' - \sin 35^\circ 20'} = \frac{35^\circ 28' 30'' - 35^\circ 20'}{35^\circ 30' - 35^\circ 20'}$$

$$= \frac{8' 30''}{10'}$$

$$= \frac{1}{2} \frac{1}{6} ;$$

$$\therefore \sin 35^\circ 28' 30'' - \sin 35^\circ 20' = \frac{1}{2} \frac{1}{6} (\sin 35^\circ 30' - \sin 35^\circ 20')$$

$$= \frac{1}{2} \frac{1}{6} \times .00237$$

$$= .00201 ;$$

$$\therefore \sin 35^\circ 28' 30'' = \sin 35^\circ 20' + .00201$$

$$= 0.57833 + .00201$$

$$= \underline{0.58034}.$$

The *form of writing out* the different parts of the example worked above is important. It should be as short and concise as possible. The student should compare the following solution, step by step, with the above, noticing especially how the decimal points are omitted.

[*Model Solution.*]

$$\left. \begin{array}{ll} \sin 35^\circ 30' & = 0.58070 \\ \sin 35^\circ 20' & = 0.57833 \end{array} \right\} \text{diff. for } 10' = +237 ;$$

$$\therefore \text{diff. for } 8' 30'' = \underline{+201} ;$$

$$\therefore \sin 35^\circ 28' 30'' = \underline{0.58034}.$$

Note. The working out of the proportional difference for 8' 30'' may be done by proportion as in the full-length solution above or by practice thus :—

$$\begin{array}{rcl} \text{diff. for } 10' & = & 237 \\ \therefore \text{diff. for } 8' & = & 189.6 \\ \text{diff. for } 30'' & = & 11.85 \\ \therefore \text{for } 8' 30'' & = & 201.45 \end{array}$$

41. Find $\cos^{-1} 0.32761 = x$ say.

It is found from the tables that

$$\cos 70^{\circ} 50' = 0.32832,$$

$$\cos 71^{\circ} 0' = 0.32557,$$

$$\text{and } \cos x = 0.32761;$$

$$\therefore \frac{\cos x - \cos 70^{\circ} 50'}{\cos 71^{\circ} 0' - \cos 70^{\circ} 50'} = \frac{-0.0071}{-0.0275} = \frac{71}{275}.$$

But, by the "Rule of Proportional Differences,"

$$\frac{\cos x - \cos 70^{\circ} 50'}{\cos 71^{\circ} 0' - \cos 70^{\circ} 50'} = \frac{x - 70^{\circ} 50'}{10'};$$

$$\therefore \frac{x - 70^{\circ} 50'}{10'} = \frac{71}{275};$$

$$\begin{aligned}\therefore x &= 70^{\circ} 50' + \frac{71}{275} \times 10' \\ &= \underline{70^{\circ} 52' 35''}.\end{aligned}$$

[Model Solution.]

$$\begin{aligned}\cos x &= 0.32761 \\ \cos 70^{\circ} 50' &= 0.32832 \\ \cos 71^{\circ} 0' &= 0.32557\end{aligned} \left\{ \begin{array}{l} \text{diff.} = -71, \\ \text{diff. for } 10' = -275; \end{array} \right.$$

$$\begin{aligned}\therefore x &= 70^{\circ} 50' + \frac{71}{275} \times 10' \\ &= \underline{70^{\circ} 52' 35''}.\end{aligned}$$

42. Find $x = \tan^{-1} 2$.

[Model Solution.]

$$\begin{aligned}\tan x &= 2.00000 \\ \tan 63^{\circ} 20' &= 1.99117 \\ \tan 63^{\circ} 30' &= 2.00569\end{aligned} \left\{ \begin{array}{l} \text{diff.} = +883, \\ \text{diff. for } 10' = +1452; \end{array} \right.$$

$$\begin{aligned}\therefore x &= 63^{\circ} 20' + \frac{883}{1452} \times 10' \\ &= \underline{63^{\circ} 26' 5''}.\end{aligned}$$

43. Find $\cot 63^\circ 26' 5''$.

[Model Solution]

$$\left. \begin{array}{l} \cot 63^\circ 30' = 0.49858 \\ \cot 63^\circ 20' = 0.50222 \end{array} \right\} \text{diff for } 10' = -364;$$

$$\therefore \text{diff. for } 6' 5'' = \frac{-222}{10};$$

$$\therefore \cot 63^\circ 26' 5'' = \underline{0.50000}$$

EXERCISES

1 Find the values of the

- (a) sine and tangent of $11^\circ 2' 37''$, $39^\circ 21' 26''$,
- (b) cosine and cosecant of $55^\circ 27' 1''$, $82^\circ 7' 28''$,
- (c) secant of $0^\circ 2' 37''$, $19^\circ 21' 37''$;
- (d) cotangent of $61^\circ 20' 20''$, $79^\circ 13' 4''$

2 Find the values of

- (a) $\sin^{-1} \frac{1}{2}$, $\sin^{-1} \frac{\sqrt{3}}{2}$, $\sin^{-1} \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\sin^{-1} \frac{1}{\sqrt{2}}$, $\sin^{-1} \frac{\sqrt{5}-1}{4}$;
- (b) $\sin^{-1} \frac{1}{4}$, $\sin^{-1} \frac{1}{3}$, $\cos^{-1} \frac{1}{5}$, $\cos^{-1} \frac{\sqrt{2}}{\sqrt{3}}$,
- (c) $\tan^{-1} 3$, $\cot^{-1} \sqrt{3}$, $\sec^{-1} 2$, $\operatorname{cosec}^{-1} 3$

3. Shew from the tables that

- (1) $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = 45^\circ$;
- (2) $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99} = 45^\circ$;
- (3) $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = 45^\circ$;
- (4) $\cos^{-1} \frac{4}{5} - \sin^{-1} \frac{1}{\sqrt{10}} + \tan^{-1} \frac{1}{2} = 45^\circ$;
- (5) $\sin^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{3\sqrt{11}} + \sin^{-1} \frac{3}{\sqrt{11}} = 90^\circ$.

§ 12. Trigonometrical Ratios for an Obtuse Angle.

44. Draw a triangle ABC in which \angle s A and B are acute.

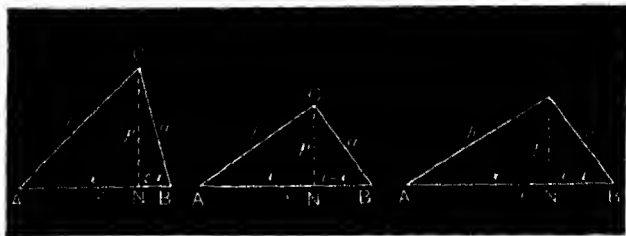


FIG. 41.

From C draw CN perpendicular to AB. Let the sides opposite the angles A, B, C be denoted by a, b, c respectively, CN by p , and area of $\triangle ABC$ by Δ .

$$\text{Then} \quad \Delta = \frac{1}{2}pc$$

Hence shew that

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B.$$

The formulae obtained in Expt. 44 may be expressed in words thus:

The area of a triangle is measured by half the product of the lengths of two sides and the sine of the included angle:

so long as the included angle is *acute*.

45. If ABC is a triangle in which C is an obtuse angle shew that its area is given by $\frac{1}{2}ab \cdot \sin (180^\circ - A)$.

46. In Fig. 42 let AN be denoted by x , then $NB = c - x$.
Shew that

$$b^2 - x^2 = a^2 - (c - x)^2,$$

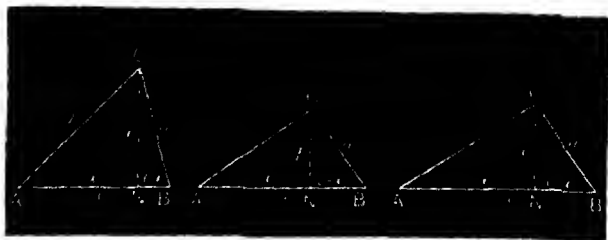


FIG. 42.

and obtain the formulae

$$b^2 + c^2 - a^2 = 2bc \cos A.$$

$$c^2 + a^2 - b^2 = 2ca \cos B.$$

The formulae obtained in Expt. 46 may be expressed in words thus:

The excess of the sum of the squares of the lengths of two sides of a triangle over the square of the length of the third side is equal to twice the product of the lengths of the two sides multiplied by the cosine of the included angle:

so long as the included angle is *acute*.

47. If ABC is a triangle in which C is an obtuse angle, shew that

$$c^2 - a^2 - b^2 = 2ab \cos(180^\circ - C).$$

It is convenient to *assume* that the formulae obtained by Expts. 44, 46 are true whether the included angle is acute, a right angle or obtuse.

In order to do this the definitions of the trigonometrical ratios must be extended, so that, if A is an obtuse angle,

$$\sin A = \sin (180^\circ - A), \quad [\text{Why?}]$$

$$\cos A = -\cos (180^\circ - A), \quad [\text{Why?}]$$

and $\sin 90^\circ = 1, \quad \cos 90^\circ = 0. \quad [\text{Why?}]$

For the present these relations will be taken as definitions, the general definitions for angles of any magnitude being given later on.

EXERCISES.

1. Deduce from Expt. 44 that the sides of any triangle are proportional to the sines of the opposite angles, i.e. shew that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}.$$

2. Prove, geometrically or otherwise, that in any triangle

$$a = b \cos C + c \cos B,$$

$$b = c \cos A + a \cos C,$$

$$c = a \cos B + b \cos A.$$

3. Deduce from the formulae of Ex. 2 that in any triangle

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

4. Shew that

$$\sin A = \frac{\sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4}}{2bc},$$

and, hence,

$$\Delta = \frac{1}{4} \sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4}.$$

5. If A, B, C are the angles of a triangle, shew that

$$\sin(A+B) = \sin C = \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = -\cos C = \cos A \cos B - \sin A \sin B,$$

by substituting the values obtained in Exs. 3, 4.

6. Verify the formula of Ex. 1 for a triangle whose sides are $a=17$, $b=44$, $c=39$; and find its area.

$$[\text{From Expt. 44, } \cos A = \frac{44^2 + 39^2 - 17^2}{2 \cdot 44 \cdot 39} = \frac{12}{13};$$

$$\therefore \sin A = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}.$$

$$\text{Similarly, } \sin B = \frac{22}{11},$$

$$\sin C = \frac{15}{17};$$

$$\begin{aligned} \therefore \sin A : \sin B : \sin C &= \frac{5}{13} : \frac{22}{11} : \frac{15}{17} \\ &= 17 : 44 : 39 \\ &= a : b : c. \end{aligned}$$

$$\text{Also } \Delta = \frac{1}{2} ab \sin C$$

$$\begin{aligned} &= \frac{1}{2} \cdot 17 \cdot 44 \cdot \frac{15}{17} \\ &= 330 \quad] \end{aligned}$$

7. Find the areas of the triangles whose sides are as follows, stating in each case which, if any, is the obtuse angle:

$$(1) \quad a=25, \quad b=26, \quad c=17;$$

$$(2) \quad a=50, \quad b=80, \quad c=78;$$

$$(3) \quad a=15, \quad b=26, \quad c=37.$$

8. A triangular field has its sides respectively 242, 1212 and 1450 yards long: shew that its area is 6 acres.

9. Assuming the "Triangle of Forces" for three forces in equilibrium, shew that

"Each force is proportional to the sine of the angle between the other two" [Lami's Theorem].

§ 13. "Solution of Triangles."

Hitherto, when solving problems on "Heights and Distances" by two observations, the triangle drawn to represent the given conditions of the problem has been solved by drawing a perpendicular breaking up the triangle into the sum or difference of two *right-angled triangles*, and the parts of these have been calculated by means of the tables on pp. 36—41, and hence the parts of the original triangle have been deduced.

The formulæ obtained in § 12 can, however, be used to solve the triangle, if it is carefully remembered that in the case of an obtuse angle X ,

$$\begin{aligned}\sin X &= \sin(180^\circ - X), \\ \cos X &= -\cos(180^\circ - X).\end{aligned}$$

Formula I. ("The rule of sines.")

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This formula can be used when any *one* fraction is completely known—say, A and a are given—and one other part, either an angle, as B , or a side as b , is also known.

Note. It will be found, as a rule, that, if an attempt is made to draw a diagram to scale, a figure is obtained (Fig. 43 *a*), which is useless for *practical purposes*, and not clear enough for illustrating a theoretical solution of the problem. Hence generally for *theoretical solution* of problems, a distorted diagram (Fig. 43 *b*) is drawn.

A. Two angles and one side.

48. A man-of-war has two big guns, one fore and the other aft. A torpedo-boat is sighted simultaneously by the captains of the guns, the telescopes making angles of $23^{\circ}37'$ and $24^{\circ}3'$ with the line from stem to stern. Given that the distance between the guns is 250 ft., find the correct range for each gun.

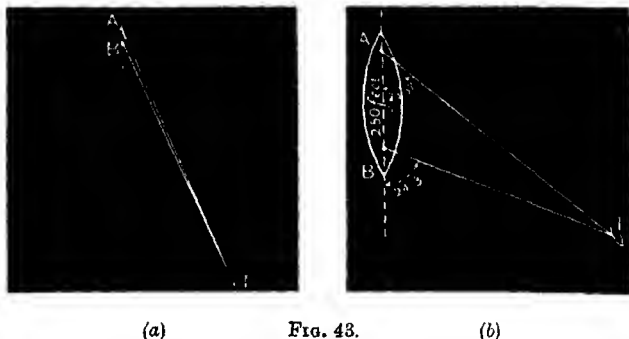


FIG. 43.

In the triangle ATB, we know two angles (and therefore the third) and one side: and we have also that

$$\sin TBA = \sin (180^{\circ} - 24^{\circ}3') = \sin 24^{\circ}3',$$

$$\text{and } \sin ATB = \sin (24^{\circ}3' - 23^{\circ}37') = \sin 26'.$$

Hence, by Formula I,

$$\frac{\sin 26'}{250} = \frac{\sin 24^{\circ}3'}{AT} = \frac{\sin 23^{\circ}37'}{BT},$$

$$\therefore AT = \frac{250 \times \sin 24^{\circ}3'}{3 \times \sin 26'} \text{ yds.} = 4492 \text{ yds.},$$

$$BT = \frac{250 \times \sin 23^{\circ}37'}{3 \times \sin 26'} \text{ yds.} = 4416 \text{ yds.}$$

Note. The average distance from any part of the ship is thus 4450 yds. The above solution is the fundamental idea of Lord Kelvin's patent range-finder; the telescopes are connected with a Wheatstone bridge arrangement and automatically register the average range on a dial in the conning-tower.

B. Two sides and the angle opposite one of them.

49. A cyclist is touring in a district for which he has a very imperfect map. The chief towns of the district A, B and C are connected by straight roads, and B and C bear E.N.E. and due E. from A. He rides from A to B and on to C, arriving at 2 p.m.; from the cyclometer he finds it is 25 miles from A to B, and 12 miles from B to C. At what time can he get back to A, riding at an average speed of 10 miles an hour?

In this problem we have

$$A = 22^\circ 30', \quad a = 12 \text{ miles}, \quad c = 25 \text{ miles}.$$

Hence, by Formula I,

$$\frac{\sin 22^\circ 30'}{12} = \frac{\sin C}{25} = \frac{\sin B}{AC},$$

$$\therefore \sin C = \frac{2}{5} \times \sin 22^\circ 30' = \sin 52^\circ 52'.$$

Since $\sin X = \sin(180 - X)$, it cannot be determined from the above calculation whether $C = 52^\circ 52'$ or $127^\circ 8'$.

$$(i) \text{ If } C = 52^\circ 52', \quad B = 180^\circ - 52^\circ 52' - 22^\circ 30' = 104^\circ 38',$$

$$\begin{aligned} \therefore AC &= \frac{12 \times \sin 104^\circ 38'}{\sin 22^\circ 30'} \text{ miles} \\ &= \frac{12 \times \sin 75^\circ 22'}{\sin 22^\circ 30'} \text{ miles} \\ &= 30.34 \text{ miles,} \end{aligned}$$

and the cyclist will get back to A about 5.2 p.m.

$$(ii) \text{ If } C = 127^\circ 8', \quad B = 180^\circ - 127^\circ 8' - 22^\circ 30' = 30^\circ 22';$$

$$\begin{aligned} \therefore AC &= \frac{12 \times \sin 30^\circ 22'}{\sin 22^\circ 30'} \\ &= 15.85 \text{ miles,} \end{aligned}$$

and the cyclist will get back to A at 3.35 p.m.

Thus there are two positions of C, such that the conditions of the problem are satisfied. This can well be shewn by means of a diagram drawn to scale.

Lay down the line **AC** due east and west, and the line **AB** towards E.N.E., making **AB** 25 miles to some convenient scale. Then a circle, with a radius equal to 12 miles on this scale, drawn with centre **B**, will cut **AC** in the two points which represent the two possible positions of **C**.

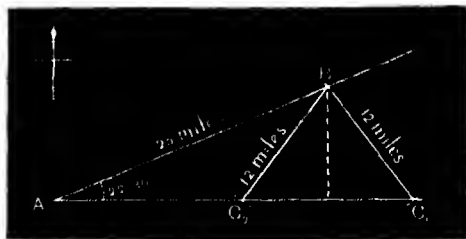


FIG. 44.

From Fig. 44 it is evident that if a is of any length greater than the perpendicular from **B** to **AC**, *i.e.* greater than $c \sin A$, the circle will intersect **AC** in two points and the solution of the problem will be ambiguous.



FIG. 45.

If, however, $a < c$, as in Fig. 45, one of these points (**C**₁) will be on the left-hand side of **A**, and the triangle **ABC**₂ will not contain the angle $22^\circ 30'$, the angle at **A** being $180^\circ - 22^\circ 30' = 157^\circ 30'$.

In the trigonometrical solution this case is determined by the fact that of the two values for C that are found from the value of $\sin C$, the sum of one of them and the given angle A is not less than 180° , and thus no triangle is possible having these angles as two of its angles.

50. In Expt. 49 suppose the cyclist to have found that BC was 31 miles instead of 12 miles. Find his time of return to A as before.

We have

$$\sin C = \frac{31}{12} \times \sin 22^\circ 30' = \sin 17^\circ 59';$$

$$\therefore C \text{ is either } 17^\circ 59' \text{ or } 162^\circ 1'.$$

The sum of A and C is either $40^\circ 29'$ or $184^\circ 31'$, *i.e.* there is only one value of B ; and we have

$$\sin B = \sin (180^\circ - 40^\circ 29') = \sin 40^\circ 29'.$$

$$\therefore AC = \frac{31 \times \sin 40^\circ 29'}{\sin 22^\circ 30'}$$

$$= 52.59 \text{ miles,}$$

and the cyclist would get back to A at 7.16 p.m.

This case is of little practical importance, but the first case is extremely important.

In survey work a "base line," as long as can conveniently be obtained is measured with extreme accuracy. The base line for English Survey on Salisbury Plain is, for instance, over 7 miles long, and the error is not more than 3 in. From the ends of this base line the bearings of another point are accurately measured with a theodolite and the triangle is solved: this gives two more lines which may be used as base-lines to find the positions of other points, and thus step by step the whole country can be covered with a series of accurately surveyed triangles, and maps can be drawn on a reduced scale.

Formula II. (*"The cosine rule."*)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \dots\dots\dots(1),$$

or $a^2 = b^2 + c^2 - 2bc \cos A \dots\dots(2).$

This formula in its first form can be used to find the angles of a triangle when the lengths of the three sides are given; in the second form, when the lengths of two sides and the magnitude of the included angle are given the third side can be found, and then the other angles can be calculated by "the rule of sines."

C. Two sides and the contained angle.

51. Two forces of 7 and 19 lbs. wt. act on a body, the angle between their directions being 43° . Find the magnitude of their resultant.

From the "vector law" it is known that, if OA, OB are drawn so that OA=7, OB=19 on some convenient scale and $\angle AOB = 43^\circ$, the resultant is represented by OC the diagonal of the parallelogram OACB.

Hence, in the triangle OAC,

$$OA=19, \quad AC=7, \quad \angle OAC=180^\circ-43^\circ.$$

Therefore, by Formula II,

$$\begin{aligned} OC^2 &= OA^2 + AC^2 - 2OA \cdot AC \cos(180^\circ - 43^\circ) \\ &= OA^2 + AC^2 + 2OA \cdot AC \cos 43^\circ \\ &= 19^2 + 7^2 + 2 \cdot 19 \cdot 7 \times (.73135) \\ &= 361 + 49 + 194.5391 \\ &= 600.5391; \end{aligned}$$

$$\therefore OC = 24.5.$$

• \therefore Resultant is a force of $24\frac{1}{2}$ lbs. wt.

D. Three sides.

52. The lengths of the sides of a triangular tract of land are 102, 195, 279 chains respectively. Calculate the magnitude of the greatest angle, and thence, by means of the formula

$$\Delta = \frac{1}{2}ab \sin C,$$

find the area of the land.

Let $a=102$, $b=195$, $c=279$ chains respectively.

Then, by Formula II,

$$\begin{aligned}\cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{102^2 + 195^2 - 279^2}{2 \cdot 102 \cdot 195} \\ &= -.73936.\end{aligned}$$

From the tables $\cos 42^\circ 19' 23'' = .73936$,

$$\therefore C = 180^\circ - 42^\circ 19' 23'' = 137^\circ 40' 37''.$$

Again,

$$\begin{aligned}\sin C &= \sin 42^\circ 19' 23'' \\ &= .67331 \text{ from the tables;}\end{aligned}$$

$$\begin{aligned}\therefore \text{area of land} &= \frac{102 \times 195 \times .67331}{2} \text{ sq. chains} \\ &= 669.6 \text{ acres.}\end{aligned}$$

EXERCISES.

1. Draw an equilateral triangle ABC ; draw AD perpendicular to BC : shew that the sides of the triangle ABD are in the ratio $1 : 2 : \sqrt{3}$. Hence prove

$$(1) \quad \sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad (3) \quad \tan 60^\circ = \sqrt{3},$$

$$(2) \quad \cos 60^\circ = \sin 30^\circ = \frac{1}{2}, \quad (4) \quad \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

2. The sides of a triangle are $2\sqrt{2}$, $2\sqrt{3}$, $\sqrt{6} - \sqrt{2}$: shew that its angles are 120° , 45° and 15° . Hence find $\sin 15^\circ$, $\cos 15^\circ$, $\tan 15^\circ$, $\sin 75^\circ$, $\cos 75^\circ$, $\tan 75^\circ$ in the form of surds.

3. In a triangle ABC, $AB=AC=2$, $BC=\sqrt{5}-1$: along BA mark off $BD=3-\sqrt{5}$, and apply the "sine rule" to shew that the triangles ABC, CBD are equiangular: hence shew that $\angle B=\angle C=72^\circ$ and $A=36^\circ$ and, using the "cosine rule," shew that

$$(1) \quad \cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4},$$

$$(2) \quad \cos 36^\circ = \sin 54^\circ = \frac{\sqrt{5}+1}{4}.$$

Exercises 4—10 should be worked without the use of the tables on pp. 36—39.

4. The sides of a triangle are as $2 : \sqrt{6} : 1 + \sqrt{3}$, find the angles.

5. The sides of a triangle are as $7 : 8 : 13$, find the greatest angle.

6. Given $C=18^\circ$, $a=\sqrt{5}+1$, $c=\sqrt{5}-1$, solve the triangle.

7. Given $A=75^\circ$, $B=30^\circ$, $b=2\sqrt{2}$, solve the triangle.

8. Given $B=30^\circ$, $c=150$, $b=50\sqrt{3}$, shew that, of the two triangles which satisfy these data, one will be isosceles and the other right-angled: find the length of the equal sides of the isosceles triangle.

9. Given $B=15^\circ$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$, solve the triangle.

10. The cosines of two of the angles of a triangle are respectively $\frac{1}{2}$ and $\frac{3}{4}$; find the ratio of the sides.

The following exercises are intended to be worked with the use of the tables on pp. 36—39.

11. The sides of a triangle are 17, 20, 27. Find all the angles.

12. Find the area of the triangle in Ex. 11, by means of the formula of Ex. 1, p. 62.

13. The angles of a triangle are in the ratio of $36 : 49 : 64$: the smallest side is 6 inches long, find the other two.

14. One side of a triangle is 153 ft. long, a second side is 69 ft. long, and the angle opposite this side is $22^{\circ}17'20''$: find the difference between the two values which are obtained for the third side in the two triangles that can be drawn to satisfy these data.

15. Two forces P , Q are balanced by a third force R : $P=27$ lbs. wt., $Q=33$ lbs. wt., $R=56$ lbs. wt. Find the angle between P and Q .

16. Two posts A and B , 500 yds. apart, are set up on the sea-shore near a fort. A man-of-war (M) appears out at sea. The angles MAB , MBA are observed to be $76^{\circ}22'30''$ and $102^{\circ}25'45''$. Find the distance of the ship from each post.

17. Two sides of a triangle are 11 and 12 and the included angle is $45^{\circ}48'56''$. Find the length of the median bisecting the third side.

18. I stand on the bank of a river directly opposite a tree on the other bank, the elevation of the top of which is $25^{\circ}30'$. I walk off at right angles to the line of sight along the bank of the river for 20 yds. when the elevation of the top of the tree becomes $24^{\circ}45'$. Find the width of the river.

19. If m is the length of the median bisecting the side BC of a triangle ABC , shew that

$$4m^2 = 2b^2 + 2c^2 - a^2.$$

20. At each end of a horizontal line, 500 yds. long, the elevations of the top of a distant hill are 35° and 37° , whilst the elevation at the middle point of the line is $36^{\circ}15'$. Find the height of the hill.

§ 14. Tables of "Indices."

All calculations mainly involving multiplications and divisions can be much simplified by the use of a table of "indices" of the powers of some chosen number.

DEF. A power of a number is the continued product of two or more factors each equal to the given number.

The number of factors in the continued product is indicated by a small figure placed above and to the right of the number—in "the postage-stamp corner."

This small figure is variously called an **exponent**, an **index**, or a **logarithm**.

53. Plot a series of points to shew the values of $y=(1.2)^x$ for $x=1, 2, 3, \dots 10$.

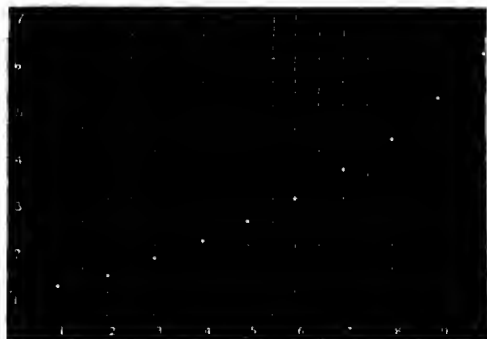


FIG. 46.

The "graph" consists of a number of isolated points corresponding to the *integral* values assigned to x ; such an expression as $(1.2)^{\frac{1}{2}}$ having no meaning according to the definition given above.

54. Join the isolated points obtained for the graph of $y=(1.2)^x$ for integral values of x with a freehand curve.

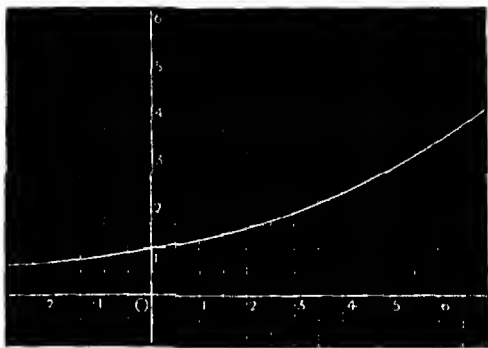


FIG. 47.

The intermediate points on this curve give us fractional, and points on its continuation to the left negative, values of x , although the above definition of an index gives us no meaning for the values of y for these points as 'powers' of 1.2. However, as the curve is *continuous*, it is most probably the locus of a point moving according to some Geometrical or Algebraical law, and, if this law can be found, it may be possible to assign meanings to the values of y when x is not an integer.

It follows at once from the given definition of a power that if a^m and a^n are two powers of a , then

$$\begin{aligned} a^m \times a^n &= (a \times a \times a \times \dots m \text{ factors}) \\ &\quad \times (a \times a \times a \times \dots n \text{ factors}) \\ &= a \times a \times a \times \dots \overbrace{m+n \text{ factors}} \\ &= a^{m+n}. \end{aligned}$$

Or graphically,

"If (x_1, y_1) , (x_2, y_2) are two points on the graph of powers of a , where x_1, x_2 are integers, then $(x_1 + x_2, y_1 \times y_2)$ is another point on the graph."

55. Take points on the graph of $y=(1.2)^x$, (drawn as in Expt. 54, but on as large a scale as possible), for which x is not a whole number; verify that, for any values of x_1, x_2 positive or negative, integral or fractional, such as those in the table below, if $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are points on the curve such that $x_3 = x_1 + x_2$, then $y_3 = y_1 y_2$.

| | | | | | | | | | | | |
|-------------|-----|-----|-----|-----|-----|-----|------|------|------|------|-----|
| $x_1 =$ | 1.2 | .3 | 2.1 | .9 | .4 | 3.6 | -4.0 | -.9 | -2.3 | -7.2 | -10 |
| $x_2 =$ | 1.3 | 1.7 | 2.6 | 7.3 | .9 | 5.2 | -2.8 | -3.2 | 9.7 | 0 | 10 |
| $x_3 =$ | 2.5 | 2.0 | 4.7 | 8.2 | 1.3 | 8.8 | -6.8 | -4.1 | 7.4 | -7.2 | 0 |
| $y_1 =$ | | | | | | | | | | | |
| $y_2 =$ | | | | | | | | | | | |
| $y_3 =$ | | | | | | | | | | | |
| $y_1 y_2 =$ | | | | | | | | | | | |

56. Assume that this "Index Law," i.e. $a^m \times a^n = a^{m+n}$ is true not only for integral values of m and n , but for all values, positive and negative, as in the case of the graphical equivalent in Expt. 55; find a meaning for $a^{\frac{1}{n}}$.

Let $a^{\frac{1}{n}} = x$.

Then $x^n = x \times x \times x \times \dots n \text{ factors,}$

$$= a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \dots n \text{ factors,}$$

$$= a^{\frac{1}{n} + \frac{1}{n} + \dots n \text{ terms,}}$$

$$= a^{\frac{n}{n}} = a,$$

$$\therefore x = \sqrt[n]{a},$$

i.e. $a^{\frac{1}{n}}$ is a symbol standing for $\sqrt[n]{a}$.

57. Find a meaning for the symbol $a^{\frac{p}{q}}$, shewing that it can be written in either of the forms

$$\sqrt[q]{a^p} \text{ or } (\sqrt[q]{a})^p.$$

58. Find a meaning for the symbol a^0 .

[We have $a^n \times a^0 = a^{n+0} = a^n$, hence a^0 stands for 1.]

59. Find a meaning for the symbol a^{-n} , where n is integral or fractional.

[We have $a^n \times a^{-n} = a^{n-n} = a^0 = 1$, hence a^{-n} stands for $1 \div a^n$.]

60. Plot the graph of $y = 10^x$, as follows :—

(i) Find $10^{\frac{1}{2}}$ ($\sqrt{10}$), $10^{\frac{1}{4}}$ ($\sqrt[4]{10}$), $10^{\frac{3}{4}}$ ($\sqrt[4]{10^3}$), $10^{\frac{1}{8}}$ ($\sqrt[8]{10}$);

(ii) Obtain $10^{\frac{1}{16}}$, $10^{\frac{3}{16}}$, $10^{\frac{5}{16}}$, $10^{\frac{7}{16}}$, and higher powers by decimal approximation.

(iii) Plot on a large sheet of squared paper, taking $1'' = 1$ along the y -axis, and $10'' = 1$ along the x -axis.

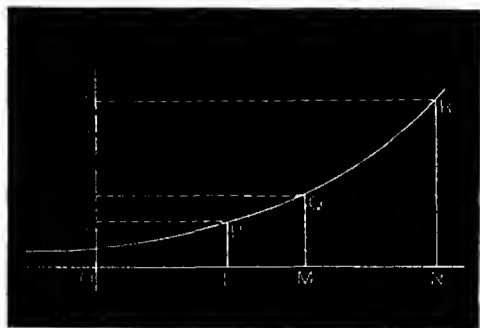


FIG. 48.

61. Use the graph of $y = 10^x$ to multiply 3.7 by 2.4 .

Find points P and Q on the curve of which the ordinates PL and QM are 2.4 and 3.7 . Along Ox mark off MN towards the right $= OL$, so that $ON = OL + OM$; draw the ordinate RN and measure it.

Verify that

$$RN = 3.7 \times 2.4.$$

62. Use the graph of $y=10^x$ to obtain the quotients

$$\frac{3.7}{2.4} \text{ and } \frac{2.4}{3.7}.$$

[For $\frac{3.7}{2.4}$, along OX mark off MN, towards the *left*, equal to OL so that $ON=OM-OL$.]

63. Obtain from the graph of $y=10^x$ the values of 10^{-1} , 10^{-2} , 10^{-3} , 10^{-4} ... 10^{-9} ; and find, by the method of interpolation explained in § 11, the values of

$$10^{-07}, 10^{-14}, 10^{-21}, 10^{-35}, 10^{-49}, 10^{-56}, 10^{-63}, 10^{-77}, 10^{-84}, 10^{-98};$$

plot the corresponding points in connection with the curve and, by noticing how near to the curve the points lie, observe for what values of x and to what extent the "Rule of proportional differences" is to be trusted with this graph or *with a corresponding set of tables*.

64. Draw as much as possible of a graph of $y=10^x$ on a large sheet of paper, for values of x between 2.2 and 2.4 taking $10''=1$ along the x axis and $1''=1$ along the y axis, and examine to what extent the "Rule of proportional differences" applies for these values of x .

1. Find from the graph of $y=10^x$ the values of

$$(a) \frac{5.3 \times 2.7}{8.6}, \quad (b) \frac{.76}{1.03 \times 2.42},$$

$$(c) (1.33)^3, \quad (d) (2.17)^2.$$

2. Extract the square roots of 3, 5, 6, 7; i.e., find values of $3^{\frac{1}{2}}$, $5^{\frac{1}{2}}$, $6^{\frac{1}{2}}$, $7^{\frac{1}{2}}$.

3. Extract the cube roots of 2, 4, 9.

§ 15. Tables of Logarithms.

It appears from the results of Expts. 63, 64, that *provided the differences are small compared with the values of x chosen*, the rule of Proportional differences is applicable to the graph of $y = 10^x$, or to the corresponding tables.

By methods of more advanced Trigonometry (*see Lock's Elementary Trigonometry* (1903), pp. 246—250), tables can be constructed, accurate to any required number of decimal places, giving the values of the index or logarithm x for consecutive values of y . A set of such tables calculated to five places of decimals is given on pp. 89, 90.

When using the tables, the student should always bear in mind that a logarithm is an index.

DEF. (a). The logarithm of a number N is the index of that power of some number a which is equal to N ; the number a is called the base of the system of logarithms,

or making use of symbols,

DEF. (b). If $a^l = N$, then l is the logarithm of N to the base a . The logarithm of N to the base a is usually written $\log_a N$.

Thus, $a^{\log_a N} = N$.

Hence it is obvious that the laws of indices apply to logarithms; in fact, the laws should be called the "laws of logarithms," the name "index" being retained as a special name for a logarithm which happens to be an integer.

65. Given $\log_a m \equiv x$, $\log_a n \equiv y$; find $\log_a (m \times n) \equiv z$

By Def. (b) on p. 78, we have

$$a^x \equiv m, \quad a^y \equiv n, \quad a^z \equiv m \times n,$$

$$\therefore a^z = a^x \times a^y = a^{x+y},$$

$$\therefore z = x + y,$$

or, more directly,

$$a^{\log_a (m \times n)} = m \times n = a^{\log_a m} \times a^{\log_a n} = a^{\log_a m + \log_a n},$$

$$\therefore \log_a (m \times n) = \log_a m + \log_a n$$

1. Shew that

$$\log_a (p \cdot q \cdot r \cdot s \cdot t) = \log_a p + \log_a q + \log_a r + \log_a s + \log_a t.$$

2. Given $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$,

find $\log_{10} 4$, $\log_{10} 6$, $\log_{10} 8$, $\log_{10} 12$, $\log_{10} 16$, $\log_{10} 18$.

3. Shew that $\log_a (m^n) = (\log_a m) \times n$.

4. Find $\log_{10} 27$, $\log_{10} 32$, $\log_{10} 64$, $\log_{10} 81$.

5. Shew that $\log_a \sqrt[n]{m} = (\log_a m) \div n$.

6. Find $\log_{10} \sqrt{2}$, $\log_{10} \sqrt[3]{3}$, $\log_{10} \sqrt[4]{9}$, $\log_{10} \sqrt[5]{12}$.

7. Shew that $\log_a \frac{m}{n} = \log_a m - \log_a n$.

8. Shew that

$$\log_a \frac{p \cdot q \cdot r}{s \cdot t} = \log_a p + \log_a q + \log_a r - \log_a s - \log_a t.$$

9. From the given values of $\log_{10} 2$ and $\log_{10} 3$ above find the
 * logarithms to the base 10 of 5, 15, $1\frac{1}{2}$, $7\frac{1}{2}$, 1.25, 3.6, 4.8.

Shew that $\log_a a^n = n$:

d the values of $\log_{10} 1$, $\log_{10} 10$, $\log_{10} 100$, $\log_{10} 1000$.

11. Given $\log_{10} 2$ and $\log_{10} 3$ above, and that $\log_{10} 7 = .84510$, $\log_{10} 120$, $\log_{10} 126$: from these by interpolation find $\log_{10} 125$, and compare it with the value obtained from $\log_{10} 5$: examine to what extent the Rule of Proportional Differences is applicable for differences as large as this.

12. (a) Take from the tables on pp. 89, 90, $\log_{10} 100$, $\log_{10} 101$ and interpolate for 100.76 ; (b) take $\log_{10} 100.7$, $\log_{10} 100.8$ and interpolate for 100.76 : compare the results obtained in (a) and (b).

Find $\log_{10} 250$, $\log_{10} 252$ and, by interpolation, $\log_{10} 25088$: compare $\log_{10} 25088$ by putting $25088 = 2^9 \cdot 7^2 \cdot 7$, and examine to what extent the Rule of Proportional Differences is applicable here.

bear

From the answers to questions 11, 12, 13 above, it will be found that, when using five-figure logarithms, the Rule of Proportional Differences can be safely applied between 1.0000 and 9.9999 , when the differences are less than $.01$, for two more significant figures, but that between 1.00 and 1.10 the differences must be less than $.001$ for safety. Hence, the tables on pp. 89, 90 are given for four significant figures, with one extra to be calculated by interpolation, for numbers between 1.0000 and 1.1000 , and for three significant figures, with two extra to be calculated, for numbers between 1.1000 and 9.9999 .

66. Find $\log_a (N \times a^n)$, given $\log_a N$.

$$\begin{aligned}\text{We have } \log_a (N \times a^n) &= \log_a N + \log_a a^n, \\ &= \log_a N + n \log_a a,\end{aligned}$$

$$\therefore \log_a (N \times a^n) = \log_a N + n.$$

Also shew that

$$\log_a (N \div a^n) = \log_a N - n.$$

Hence

$$\log_{10} (N \times 10) = \log_{10} N + 1,$$

$$\log_{10} (N \times 10^2) = \log_{10} N + 2,$$

$$\log_{10} (N \div 10) = \log_{10} N - 1,$$

$$\log_{10} (N \div 10^2) = \log_{10} N - 2;$$

and so on.

Thus it appears that any two decimal numbers, *expressed by the same significant figures in the same order*, have logarithms to base 10 which *differ only by whole numbers*; that is they have the decimal part of their logarithms—called the **mantissa**—identical. For multiplication by 10 only moves the decimal point in the number one place to the right, leaving the sequence of significant figures unaltered; whilst it adds on $\log_{10} 10$ ($=1$) to the logarithm, thus leaving the mantissa unaltered.

Any number can be expressed as the product of a number between 1 and 10 and a power of 10. This operation is called “reducing the number to **Standard Form**.”

Thus

$$24696 = 2.4696 \times 10^4,$$

$$.0024696 = 2.4696 \times 10^{-3}.$$

When a number is reduced to Standard Form, not only is the real value of each significant figure much more easily recognized, but also its logarithm is more easily determined.

14. Given $\log_{10} 3425 = 3.53466$, find

$$\log_{10} 34.25, \log_{10} .03425.$$

67. Given $\log_{10} 2 = \cdot 30103$, $\log_{10} 3 = \cdot 47712$, $\log_{10} 7 = \cdot 84510$,
find $\log_{10} 24696$, $\log_{10} 2\cdot 4696$, $\log_{10} \cdot 0024696$.

[Since $24696 = 2^3 \cdot 3^2 \cdot 7^3$;

$$\begin{aligned}\therefore \log_{10} 24696 &= 3 \log_{10} 2 + 2 \log_{10} 3 + 3 \log_{10} 7, \\ &= 0\cdot 90309 \\ &\quad + 0\cdot 95424 \\ &\quad + \underline{2\cdot 53530} \\ &= 4\cdot 39263.\end{aligned}$$

Now $24696 = 2\cdot 4696 \times 10^4$,

$$\therefore \log 2\cdot 4696 = 0\cdot 39263.$$

Again, $\cdot 0024696 = 2\cdot 4696 \times 10^{-3}$,

$$\begin{aligned}\therefore \log \cdot 0024696 &= \log 2\cdot 4696 - 3 \\ &= 0\cdot 39263 - 3.\end{aligned}$$

It follows from Expt. 66 that, if the logarithm of $2\cdot 4696$ is known, the logarithms of all numbers having this sequence of significant figures can be written down immediately. Hence, in constructing a system of tables to the base 10 only the mantissae for the sequences of figures need be given, the integral part of the logarithm—called the **characteristic**—being attached according to the following rule.

RULE. The characteristic of the logarithm of a number is the index of the power of 10 which appears as a factor when the number is brought to standard form.

15. Express the following numbers in standard form, and find the characteristics of their logarithms:—

$$3\cdot 672, 230\cdot 75, 0\cdot 023, 0\cdot 3, 0\cdot 00001.$$

16. Find, without actually calculating the number itself, the “place-value” of the first significant figure in

$$2^{32}, 16^7, \sqrt{0\cdot 3}, \sqrt[10]{1000}.$$

The use of tables of Logarithms necessitates keeping the mantissa positive, even for the logarithms of such numbers as $\cdot 0024696$. It is inconvenient to write the logarithm of this number as $0\cdot 39263 - 3$ or $-3 + 0\cdot 39263$; whilst it is incorrect to write it as $-3\cdot 39263$, which would stand for $-3 - 0\cdot 39263$. Hence the **minus sign is written over the characteristic** to show that it alone, and not the mantissa, is negative; thus,

$$\log_{10} \cdot 0024696 = \bar{3}\cdot 39263 \text{ (read "bar three, point, etc.")}$$

This must be most carefully remembered, especially in calculating logarithms of square, cube and other *roots* of numbers.

$$68. \text{ Find } \log_{10} \sqrt{\cdot 0024696}, \log_{10} \sqrt[5]{\cdot 0024696}.$$

$$\begin{aligned} \text{[We have } \log_{10} \sqrt{\cdot 0024696} &= \frac{1}{2} \log_{10} \cdot 0024696 \\ &= \frac{1}{2} (\bar{3}\cdot 39263) \\ &= \left\{ \begin{aligned} &= \frac{1}{2} (-3 + \cdot 39263) \\ &= \frac{1}{2} (-4 + 1\cdot 39263) \\ &= -2 + 0\cdot 69632 \end{aligned} \right\} \\ &= \bar{2}\cdot 69632. \end{aligned}$$

$$\begin{aligned} \text{Again, } \log_{10} \sqrt[5]{\cdot 0024696} &= \frac{1}{5} \log_{10} \cdot 0024696 \\ &= \frac{1}{5} (\bar{3}\cdot 39263) \\ &= \left\{ \begin{aligned} &= \frac{1}{5} (-3 + 0\cdot 39263) \\ &= \frac{1}{5} (-5 + 2\cdot 39263) \\ &= -1 + 0\cdot 47853 \end{aligned} \right\} \\ &= \bar{1}\cdot 47853.] \end{aligned}$$

Note. The three lines of working enclosed by the brackets represent mental work. The negative characteristic is increased until it becomes a multiple of the divisor, and at the same time the positive mantissa is increased by the same integer; the division is then performed in two parts, and the quotients once more are associated.

Tables of the logarithms of the trigonometrical ratios can also be constructed ; and the rule of proportional differences, being true for the natural trigonometrical ratios (§ 11), and also for logarithms (§ 15), is true for the logarithms of the ratios.

Sines and cosines are always less than unity, as also are the tangents of all angles between 0° and 45° . The logarithms of these Ratios must therefore have *negative* characteristics.

To avoid the inconvenience of having to print these negative characteristics, the whole number 10 is added to each logarithm of the Trigonometrical Ratios, before it is set down in the Table. The numbers thus recorded are called the *tabular logarithms* of the sine, cosine, etc., of an angle.

Thus opposite $31^\circ 15'$ in the logarithmic tables of sines we find 9.71498,

$$\therefore \log \sin 31^\circ 15' = 9.71498 - 10 = \bar{1}.71498.$$

Tabular logarithms are indicated by the letter '*L*.' Thus $L\sin 31^\circ 15'$ stands for the tabular logarithm of $\sin 31^\circ 15'$, and is equal to $\{\log \sin 31^\circ 15' + 10\}$. For distinction $\log \sin 31^\circ 15'$ is read "log-sine" (short for logarithmic sine) whilst the tabular logarithm $L\sin 31^\circ 15'$ is read "el-sine."

As in the case of the use of tables of the natural trigonometrical functions, the form of setting out logarithmic work is important. The following general rules should be attended to.

- (a) Arrange all work in strictly logical form : do not be slipshod.
- (b) Arrange logarithms to be added *in columns*, not in rows : keep the decimal points under one another.

- (c) Do not write any unnecessary figures.
 (d) Do all calculations for interpolation on scrap-paper or set aside a column on the right-hand side of the page for the purpose.
 (e) Keep the mantissae positive.

EXERCISES.

1. Find the value of $\sqrt[3]{\frac{3}{7}}$.

[*Model Solution.*]

$$\begin{aligned}
 \log \sqrt[3]{\frac{3}{7}} &= \frac{1}{3} (\log 3 - \log 7) \\
 &= 0.15904 \\
 &\quad - 0.28170 \\
 &\quad \hline
 &= \bar{1}.87734 \left. \begin{array}{l} \text{diff.} = 54 \\ \text{diff.} = 57 \end{array} \right\} \\
 \text{From the tables, } \log 7.53 &= \begin{array}{r} 680 \\ 4 \quad 737 \end{array} \\
 \therefore \sqrt[3]{\frac{3}{7}} &= 7.53 \frac{5}{8} \times 10^{-1} \\
 &= \underline{\underline{0.75395}}.
 \end{aligned}$$

2. Find the value of $(0.0327)^3$.

[*Model Solution.*]

$$\begin{aligned}
 \log (0.0327)^3 &= 3 (\log 0.0327) \\
 &= 3 (\bar{2}.51455) \\
 &= \bar{5}.54365 \left. \begin{array}{l} \text{diff.} = 82 \\ \text{diff.} = 124 \end{array} \right\} \\
 \text{From the tables, } \log 3.49 &= \begin{array}{r} 283 \\ 50 = 407 \end{array} \\
 \therefore (0.0327)^3 &= 3.49 \frac{82}{124} \times 10^{-6} \\
 &= \underline{\underline{0.00034966}}.
 \end{aligned}$$

3. Find the value of $\alpha = \frac{24 \times \sin 72^\circ 4'}{\sin 65^\circ 59' 42''}$.

[Model Solution.]

$$\begin{aligned} \log \alpha &= \log 24 + L \sin 72^\circ 4' - L \sin 65^\circ 59' 42'' \quad (\alpha) \\ &= 1.38021 \quad - 9.96017 \quad (\delta) \\ &\quad 9.97821 \quad (\beta) \quad \underline{54 \quad (\epsilon)} \\ &\quad \underline{16 \quad (\gamma)} \\ &= 11.35858 \\ &\quad - 9.96071 \\ &= 1.39787 \end{aligned}$$

From the tables $\log 2.49 = \begin{matrix} 620 \\ 50 = 795 \end{matrix} \left. \begin{matrix} \text{diff.} = 167 \\ \text{diff.} = 175; \end{matrix} \right\}$

$$\begin{aligned} \therefore \alpha &= 2.49 \frac{167}{175} \times 10^1 \\ &= \underline{24.995}. \end{aligned}$$

4. Find A, given $\tan \frac{A}{2} = \sqrt{\frac{191.42 \times 160.83}{674.10 \times 321.85}}$.

[Model Solution.]

$$\begin{aligned} \log \left(\frac{191.42 \times 160.83}{674.10 \times 321.85} \right) &= 2.28103 \quad (\zeta) \quad - 2.82866 \quad (\kappa) \\ &\quad 95 \quad (\eta) \quad \quad \quad 6 \quad (\lambda) \\ &\quad 2.20412 \quad (\theta) \quad \quad \quad 2.50651 \quad (\mu) \\ &\quad \underline{225 \quad (\iota)} \quad \quad \quad \underline{115 \quad (\nu)} \\ &= 4.48835 \\ &\quad - 5.33638 \\ &= \bar{1}.15197 \end{aligned}$$

$$\therefore \log \tan \frac{A}{2} = \bar{1}.157599;$$

From the tables $L \tan \frac{A}{2} = \begin{matrix} 9.57599 \\ L \tan 20^\circ 30' = 274 \\ 40' = 658 \end{matrix} \left. \begin{matrix} \text{diff.} = 325 \\ \text{diff. for } 10' = 384; \end{matrix} \right\}$

$$\therefore \frac{A}{2} = 20^\circ 30' + \frac{384}{325} \times 10'$$

$$= 20^\circ 38' 28'';$$

$$\therefore A = \underline{41^\circ 16' 56''}.$$

Find the values of

5. $\frac{1}{(1.44)^3}$. 8. $0.81 \div \sqrt{112}$. 11. $(\frac{2}{3})^{\frac{1}{2}}$.
 6. $\sqrt{0.0125}$. 9. $\sqrt[5]{\frac{7}{80}}$. 12. $(0.03)^6 \div (0.12)^4$.
 7. $\frac{(1.08)^4}{(0.0147)^3}$. 10. $(1.05)^{20}$. 13. $(18)^{\sqrt{2}}$.

Find the logarithms of

14. $(\sin 18^\circ 37')^{-2}$. 16. $\sqrt[3]{(\tan 13^\circ 12' 45')}$.
 15. $(\tan 35^\circ 42')^3$. 17. $(\cos 26^\circ 33')^{-\frac{1}{2}}$.

Find the value of x , correct to three decimal places, that satisfies each of the equations:

18. $15^x = 20$. 19. $7^x = 3^{x+1} \div 2^{x-2}$.

20. If $\log \frac{a-b}{3} = \frac{1}{2}(\log a + \log b)$,
 shew that $a^2 + b^2 = 11ab$.

-
- (α) The additional 10s in the tabular logarithms cancel one another.
 (β) $L \sin 72^\circ$ from the tables. (γ) prop. diff. for $4'$.
 (δ) $L \sin 63^\circ 50'$ from the tables. (ϵ) prop. diff. for $9' 42''$.
 (ζ) $\log 191$. (η) diff. for 42. (θ) $\log 160$. (ι) diff. for 83.
 (κ) $\log 674$. (λ) diff. for 10. (μ) $\log 821$. (ν) diff. for 85.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 100 | 00000 | 00043 | 00087 | 00130 | 00173 | 00217 | 00260 | 00303 | 00346 | 00389 |
| 1 | 00432 | 00475 | 00518 | 00560 | 00604 | 00647 | 00689 | 00732 | 00775 | 00817 |
| 2 | 00860 | 00903 | 00945 | 00988 | 01030 | 01072 | 01115 | 01157 | 01199 | 01242 |
| 3 | 01284 | 01326 | 01368 | 01410 | 01452 | 01494 | 01536 | 01578 | 01620 | 01662 |
| 4 | 01703 | 01745 | 01787 | 01828 | 01870 | 01912 | 01953 | 01995 | 02036 | 02078 |
| 5 | 02119 | 02160 | 02202 | 02243 | 02284 | 02325 | 02366 | 02408 | 02449 | 02490 |
| 6 | 02531 | 02572 | 02612 | 02653 | 02694 | 02735 | 02776 | 02816 | 02857 | 02898 |
| 7 | 02938 | 02979 | 03019 | 03060 | 03100 | 03141 | 03181 | 03222 | 03262 | 03302 |
| 8 | 03342 | 03383 | 03423 | 03463 | 03503 | 03543 | 03583 | 03623 | 03663 | 03703 |
| 9 | 03743 | 03782 | 03822 | 03862 | 03902 | 03941 | 03981 | 04021 | 04060 | 04100 |
| 11 | 04139 | 04532 | 04922 | 05308 | 05690 | 06070 | 06446 | 06819 | 07188 | 07555 |
| 2 | 07918 | 08297 | 08676 | 09051 | 09422 | 09791 | 10037 | 10380 | 10721 | 11060 |
| 3 | 11394 | 11727 | 12057 | 12385 | 12710 | 13033 | 13354 | 13672 | 13988 | 14301 |
| 4 | 14613 | 14922 | 15229 | 15534 | 15836 | 16137 | 16435 | 16732 | 17026 | 17319 |
| 5 | 17609 | 17898 | 18184 | 18469 | 18752 | 19033 | 19312 | 19590 | 19866 | 20140 |
| 6 | 20412 | 20710 | 20952 | 21219 | 21484 | 21748 | 22011 | 22272 | 22531 | 22789 |
| 7 | 23045 | 23300 | 23553 | 23805 | 24055 | 24304 | 24551 | 24797 | 25042 | 25285 |
| 8 | 25527 | 25768 | 26007 | 26245 | 26482 | 26717 | 26951 | 27184 | 27416 | 27646 |
| 9 | 27875 | 28103 | 28330 | 28556 | 28780 | 29003 | 29226 | 29447 | 29667 | 29885 |
| 20 | 30103 | 30320 | 30535 | 30750 | 30963 | 31175 | 31387 | 31597 | 31806 | 32015 |
| 1 | 32222 | 32428 | 32634 | 32838 | 33041 | 33244 | 33445 | 33646 | 33846 | 34044 |
| 2 | 34242 | 34439 | 34635 | 34830 | 35025 | 35218 | 35411 | 35603 | 35793 | 35984 |
| 3 | 36173 | 36361 | 36549 | 36736 | 36922 | 37107 | 37291 | 37475 | 37658 | 37840 |
| 4 | 38021 | 38202 | 38382 | 38561 | 38739 | 38917 | 39094 | 39270 | 39445 | 39620 |
| 5 | 39794 | 39967 | 40140 | 40312 | 40483 | 40654 | 40824 | 40993 | 41162 | 41330 |
| 6 | 41497 | 41664 | 41830 | 41996 | 42160 | 42325 | 42488 | 42651 | 42813 | 42975 |
| 7 | 43136 | 43297 | 43457 | 43616 | 43775 | 43933 | 44091 | 44248 | 44404 | 44560 |
| 8 | 44716 | 44871 | 45025 | 45179 | 45332 | 45484 | 45637 | 45788 | 45939 | 46090 |
| 9 | 46240 | 46389 | 46538 | 46687 | 46835 | 46982 | 47129 | 47276 | 47422 | 47567 |
| 30 | 47712 | 47857 | 48001 | 48144 | 48287 | 48430 | 48572 | 48714 | 48855 | 48996 |
| 1 | 49136 | 49276 | 49415 | 49554 | 49693 | 49831 | 49969 | 50106 | 50243 | 50379 |
| 2 | 50515 | 50651 | 50786 | 50920 | 51055 | 51188 | 51322 | 51455 | 51587 | 51720 |
| 3 | 51857 | 51983 | 52114 | 52244 | 52375 | 52504 | 52634 | 52763 | 52892 | 53020 |
| 4 | 53148 | 53275 | 53403 | 53529 | 53656 | 53782 | 53908 | 54033 | 54158 | 54283 |
| 5 | 54407 | 54531 | 54654 | 54777 | 54900 | 55023 | 55145 | 55267 | 55388 | 55509 |
| 6 | 55630 | 55751 | 55871 | 55991 | 56110 | 56229 | 56348 | 56467 | 56585 | 56703 |
| 7 | 56820 | 56937 | 57054 | 57171 | 57287 | 57403 | 57519 | 57634 | 57749 | 57864 |
| 8 | 57978 | 58093 | 58206 | 58320 | 58433 | 58546 | 58659 | 58771 | 58883 | 58995 |
| 9 | 59106 | 59218 | 59329 | 59439 | 59550 | 59660 | 59770 | 59879 | 59988 | 60097 |
| 40 | 60206 | 60314 | 60423 | 60531 | 60638 | 60746 | 60853 | 60959 | 61066 | 61172 |
| 1 | 61278 | 61384 | 61490 | 61595 | 61700 | 61805 | 61909 | 62014 | 62118 | 62221 |
| 2 | 62325 | 62428 | 62531 | 62634 | 62737 | 62839 | 62941 | 63043 | 63144 | 63246 |
| 3 | 63347 | 63448 | 63548 | 63649 | 63749 | 63849 | 63949 | 64048 | 64147 | 64246 |
| 4 | 64345 | 64444 | 64542 | 64640 | 64738 | 64836 | 64933 | 65031 | 65128 | 65225 |
| 5 | 65321 | 65418 | 65514 | 65610 | 65706 | 65801 | 65896 | 65992 | 66087 | 66181 |
| 6 | 66276 | 66370 | 66464 | 66558 | 66652 | 66745 | 66839 | 66932 | 67025 | 67117 |
| 7 | 67210 | 67302 | 67394 | 67486 | 67578 | 67669 | 67761 | 67852 | 67943 | 68034 |
| 8 | 68124 | 68215 | 68305 | 68395 | 68485 | 68574 | 68664 | 68753 | 68842 | 68931 |
| 9 | 69020 | 69108 | 69197 | 69285 | 69373 | 69461 | 69548 | 69636 | 69723 | 69810 |
| 50 | 69897 | 69984 | 70070 | 70157 | 70243 | 70329 | 70415 | 70501 | 70586 | 70672 |

One more figure

The "Table of Proportional Parts" may be used to find two more figures

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 50 | 69897 | 69984 | 70070 | 70157 | 70243 | 70329 | 70415 | 70501 | 70586 | 70672 |
| 1 | 70757 | 70842 | 70927 | 71012 | 71096 | 71181 | 71265 | 71349 | 71433 | 71517 |
| 2 | 71600 | 71684 | 71767 | 71850 | 71933 | 72016 | 72099 | 72181 | 72263 | 72346 |
| 3 | 72428 | 72509 | 72591 | 72673 | 72754 | 72835 | 72916 | 72997 | 73078 | 73159 |
| 4 | 73239 | 73320 | 73400 | 73480 | 73560 | 73640 | 73719 | 73799 | 73878 | 73957 |
| 5 | 74036 | 74115 | 74194 | 74273 | 74351 | 74429 | 74507 | 74586 | 74663 | 74741 |
| 6 | 74819 | 74896 | 74974 | 75051 | 75128 | 75205 | 75282 | 75358 | 75435 | 75511 |
| 7 | 75587 | 75664 | 75740 | 75815 | 75891 | 75967 | 76042 | 76118 | 76193 | 76268 |
| 8 | 76343 | 76418 | 76492 | 76567 | 76641 | 76716 | 76790 | 76864 | 76938 | 77012 |
| 9 | 77085 | 77159 | 77232 | 77305 | 77379 | 77452 | 77525 | 77597 | 77670 | 77743 |
| 60 | 77815 | 77887 | 77960 | 78032 | 78104 | 78176 | 78247 | 78319 | 78390 | 78462 |
| 1 | 78533 | 78604 | 78675 | 78746 | 78817 | 78888 | 78958 | 79029 | 79099 | 79169 |
| 2 | 79239 | 79309 | 79379 | 79449 | 79518 | 79588 | 79657 | 79727 | 79796 | 79865 |
| 3 | 79934 | 80003 | 80072 | 80140 | 80209 | 80277 | 80346 | 80414 | 80482 | 80550 |
| 4 | 80618 | 80686 | 80754 | 80821 | 80889 | 80956 | 81023 | 81090 | 81158 | 81224 |
| 5 | 81291 | 81358 | 81425 | 81491 | 81558 | 81624 | 81690 | 81757 | 81823 | 81889 |
| 6 | 81954 | 82020 | 82086 | 82151 | 82217 | 82282 | 82347 | 82413 | 82478 | 82543 |
| 7 | 82607 | 82672 | 82737 | 82802 | 82866 | 82930 | 82995 | 83059 | 83123 | 83187 |
| 8 | 83251 | 83315 | 83378 | 83442 | 83506 | 83569 | 83632 | 83696 | 83759 | 83822 |
| 9 | 83885 | 83948 | 84011 | 84073 | 84136 | 84198 | 84261 | 84323 | 84386 | 84448 |
| 70 | 84510 | 84572 | 84634 | 84696 | 84757 | 84819 | 84880 | 84942 | 85003 | 85065 |
| 1 | 85126 | 85187 | 85248 | 85309 | 85370 | 85431 | 85491 | 85552 | 85612 | 85673 |
| 2 | 85733 | 85794 | 85854 | 85914 | 85974 | 86034 | 86094 | 86153 | 86213 | 86273 |
| 3 | 86332 | 86392 | 86451 | 86510 | 86570 | 86629 | 86688 | 86747 | 86806 | 86864 |
| 4 | 86923 | 86982 | 87040 | 87099 | 87157 | 87216 | 87274 | 87332 | 87390 | 87448 |
| 5 | 87506 | 87564 | 87622 | 87680 | 87737 | 87795 | 87852 | 87910 | 87967 | 88024 |
| 6 | 88081 | 88138 | 88196 | 88252 | 88309 | 88366 | 88423 | 88480 | 88536 | 88593 |
| 7 | 88649 | 88705 | 88762 | 88818 | 88874 | 88930 | 88986 | 89042 | 89098 | 89154 |
| 8 | 89209 | 89265 | 89321 | 89376 | 89432 | 89487 | 89542 | 89597 | 89653 | 89708 |
| 9 | 89763 | 89818 | 89873 | 89927 | 89982 | 90037 | 90091 | 90146 | 90200 | 90255 |
| 80 | 90309 | 90363 | 90417 | 90472 | 90526 | 90580 | 90633 | 90687 | 90741 | 90795 |
| 1 | 90849 | 90902 | 90956 | 91009 | 91062 | 91116 | 91169 | 91222 | 91275 | 91328 |
| 2 | 91381 | 91434 | 91487 | 91540 | 91593 | 91645 | 91698 | 91751 | 91803 | 91855 |
| 3 | 91908 | 91960 | 92012 | 92065 | 92117 | 92169 | 92221 | 92273 | 92324 | 92376 |
| 4 | 92428 | 92480 | 92531 | 92583 | 92634 | 92686 | 92737 | 92788 | 92840 | 92891 |
| 5 | 92942 | 92993 | 93044 | 93095 | 93146 | 93197 | 93247 | 93298 | 93349 | 93399 |
| 6 | 93450 | 93500 | 93551 | 93601 | 93651 | 93702 | 93752 | 93802 | 93852 | 93902 |
| 7 | 93952 | 94002 | 94052 | 94101 | 94151 | 94201 | 94250 | 94300 | 94349 | 94399 |
| 8 | 94448 | 94498 | 94547 | 94596 | 94645 | 94694 | 94743 | 94792 | 94841 | 94890 |
| 9 | 94939 | 94988 | 95036 | 95085 | 95134 | 95182 | 95231 | 95279 | 95328 | 95376 |
| 90 | 95424 | 95472 | 95521 | 95569 | 95617 | 95665 | 95713 | 95761 | 95809 | 95856 |
| 1 | 95904 | 95952 | 95999 | 96047 | 96095 | 96142 | 96190 | 96237 | 96284 | 96332 |
| 2 | 96379 | 96426 | 96473 | 96520 | 96567 | 96614 | 96661 | 96708 | 96755 | 96802 |
| 3 | 96848 | 96895 | 96942 | 96988 | 97035 | 97081 | 97128 | 97174 | 97220 | 97267 |
| 4 | 97313 | 97359 | 97405 | 97451 | 97497 | 97543 | 97589 | 97635 | 97681 | 97727 |
| 5 | 97772 | 97818 | 97864 | 97909 | 97955 | 98000 | 98046 | 98091 | 98137 | 98182 |
| 6 | 98227 | 98272 | 98318 | 98363 | 98408 | 98453 | 98498 | 98543 | 98588 | 98632 |
| 7 | 98677 | 98722 | 98767 | 98811 | 98856 | 98900 | 98945 | 98989 | 99034 | 99078 |
| 8 | 99123 | 99167 | 99211 | 99255 | 99300 | 99344 | 99388 | 99432 | 99476 | 99520 |
| 9 | 99564 | 99607 | 99651 | 99695 | 99739 | 99782 | 99826 | 99870 | 99913 | 99957 |

The "Rule of Proportional Parts" may be used to find two more figures.

| Sines | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|-------|---------|---------|---------|---------|---------|---------|---------|----------|
| 0° | — ∞ | 7'46373 | 7'76475 | 7'94084 | 8'06578 | 8'16268 | 8'24186 | 89° |
| 1 | 8'24186 | 8'30879 | 8'36678 | 8'41792 | 8'46366 | 8'50504 | 8'54282 | 88 |
| 2 | 8'54282 | 8'57757 | 8'60973 | 8'63968 | 8'66769 | 8'69400 | 8'71880 | 87 |
| 3 | 8'71880 | 8'74226 | 8'76451 | 8'78568 | 8'80585 | 8'82513 | 8'84358 | 86 |
| 4 | 8'84358 | 8'86128 | 8'87829 | 8'89494 | 8'91040 | 8'92561 | 8'94030 | 85 |
| 5 | 8'94030 | 8'95450 | 8'96825 | 8'98157 | 8'99450 | 9'00704 | 9'01923 | 84 |
| 6 | 9'01923 | 9'03109 | 9'04262 | 9'05386 | 9'06481 | 9'07548 | 9'08589 | 83 |
| 7 | 9'08589 | 9'09606 | 9'10599 | 9'11570 | 9'12519 | 9'13447 | 9'14356 | 82 |
| 8 | 9'14356 | 9'15245 | 9'16116 | 9'16970 | 9'17807 | 9'18628 | 9'19433 | 81 |
| 9 | 9'19433 | 9'20223 | 9'20999 | 9'21761 | 9'22509 | 9'23244 | 9'23967 | 80 |
| 10 | 9'23967 | 9'24677 | 9'25376 | 9'26063 | 9'26739 | 9'27405 | 9'28060 | 79 |
| 11 | 9'28060 | 9'28705 | 9'29340 | 9'29966 | 9'30582 | 9'31189 | 9'31788 | 78 |
| 12 | 9'31788 | 9'32378 | 9'32960 | 9'33534 | 9'34100 | 9'34658 | 9'35209 | 77 |
| 13 | 9'35209 | 9'35752 | 9'36289 | 9'36819 | 9'37341 | 9'37858 | 9'38368 | 76 |
| 14 | 9'38368 | 9'38871 | 9'39369 | 9'39860 | 9'40346 | 9'40825 | 9'41300 | 75 |
| 15 | 9'41300 | 9'41768 | 9'42232 | 9'42690 | 9'43143 | 9'43591 | 9'44034 | 74 |
| 16 | 9'44034 | 9'44472 | 9'44905 | 9'45334 | 9'45758 | 9'46178 | 9'46594 | 73 |
| 17 | 9'46594 | 9'47005 | 9'47411 | 9'47814 | 9'48213 | 9'48607 | 9'48998 | 72 |
| 18 | 9'48998 | 9'49385 | 9'49768 | 9'50148 | 9'50523 | 9'50896 | 9'51264 | 71 |
| 19 | 9'51264 | 9'51629 | 9'51991 | 9'52350 | 9'52705 | 9'53057 | 9'53405 | 70 |
| 20 | 9'53405 | 9'53751 | 9'54093 | 9'54433 | 9'54769 | 9'55102 | 9'55433 | 69 |
| 21 | 9'55433 | 9'55761 | 9'56095 | 9'56428 | 9'56757 | 9'57084 | 9'57408 | 68 |
| 22 | 9'57408 | 9'57729 | 9'58052 | 9'58368 | 9'58689 | 9'58998 | 9'59305 | 67 |
| 23 | 9'59305 | 9'59618 | 9'59928 | 9'60234 | 9'60536 | 9'60836 | 9'61133 | 66 |
| 24 | 9'61133 | 9'61434 | 9'61733 | 9'62029 | 9'62323 | 9'62614 | 9'62902 | 65 |
| 25 | 9'62902 | 9'63199 | 9'63493 | 9'63786 | 9'64076 | 9'64363 | 9'64648 | 64 |
| 26 | 9'64648 | 9'64934 | 9'65218 | 9'65500 | 9'65780 | 9'66057 | 9'66332 | 63 |
| 27 | 9'66332 | 9'66605 | 9'66876 | 9'67145 | 9'67412 | 9'67677 | 9'67940 | 62 |
| 28 | 9'67940 | 9'68200 | 9'68458 | 9'68714 | 9'68968 | 9'69219 | 9'69468 | 61 |
| 29 | 9'69468 | 9'69715 | 9'69960 | 9'70204 | 9'70446 | 9'70687 | 9'70926 | 60 |
| 30 | 9'70926 | 9'71163 | 9'71398 | 9'71631 | 9'71862 | 9'72091 | 9'72318 | 59 |
| 31 | 9'72318 | 9'72544 | 9'72768 | 9'72990 | 9'73210 | 9'73428 | 9'73644 | 58 |
| 32 | 9'73644 | 9'73859 | 9'74073 | 9'74285 | 9'74495 | 9'74703 | 9'74909 | 57 |
| 33 | 9'74909 | 9'75113 | 9'75315 | 9'75516 | 9'75715 | 9'75912 | 9'76108 | 56 |
| 34 | 9'76108 | 9'76303 | 9'76496 | 9'76687 | 9'76876 | 9'77063 | 9'77248 | 55 |
| 35 | 9'77248 | 9'77434 | 9'77618 | 9'77800 | 9'77980 | 9'78158 | 9'78334 | 54 |
| 36 | 9'78334 | 9'78509 | 9'78682 | 9'78853 | 9'79022 | 9'79189 | 9'79354 | 53 |
| 37 | 9'79354 | 9'79519 | 9'79682 | 9'79843 | 9'79999 | 9'80154 | 9'80308 | 52 |
| 38 | 9'80308 | 9'80461 | 9'80612 | 9'80762 | 9'80910 | 9'81056 | 9'81200 | 51 |
| 39 | 9'81200 | 9'81343 | 9'81484 | 9'81623 | 9'81760 | 9'81895 | 9'82029 | 50 |
| 40 | 9'82029 | 9'82163 | 9'82295 | 9'82426 | 9'82555 | 9'82682 | 9'82808 | 49 |
| 41 | 9'82808 | 9'82934 | 9'83059 | 9'83183 | 9'83305 | 9'83426 | 9'83546 | 48 |
| 42 | 9'83546 | 9'83663 | 9'83779 | 9'83893 | 9'84006 | 9'84118 | 9'84229 | 47 |
| 43 | 9'84229 | 9'84343 | 9'84455 | 9'84566 | 9'84674 | 9'84781 | 9'84887 | 46 |
| 44 | 9'84887 | 9'84991 | 9'85094 | 9'85196 | 9'85296 | 9'85394 | 9'85491 | 45 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Co-sines |

| Sines | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|-------|---------|---------|---------|---------|---------|---------|----------|---------|
| 45° | 9'84949 | 9'85074 | 9'85200 | 9'85324 | 9'85448 | 9'85571 | 9'85693 | 44° |
| 46 | 9'85693 | 9'85815 | 9'85936 | 9'86056 | 9'86176 | 9'86295 | 9'86413 | 43 |
| 47 | 9'86413 | 9'86530 | 9'86647 | 9'86763 | 9'86879 | 9'86993 | 9'87107 | 42 |
| 48 | 9'87107 | 9'87221 | 9'87334 | 9'87446 | 9'87557 | 9'87668 | 9'87778 | 41 |
| 49 | 9'87778 | 9'87887 | 9'87996 | 9'88105 | 9'88212 | 9'88319 | 9'88425 | 40 |
| 50 | 9'88425 | 9'88531 | 9'88636 | 9'88741 | 9'88844 | 9'88948 | 9'89050 | 39 |
| 51 | 9'89050 | 9'89152 | 9'89254 | 9'89354 | 9'89455 | 9'89554 | 9'89653 | 38 |
| 52 | 9'89653 | 9'89752 | 9'89849 | 9'89947 | 9'90043 | 9'90139 | 9'90235 | 37 |
| 53 | 9'90235 | 9'90330 | 9'90424 | 9'90518 | 9'90611 | 9'90704 | 9'90796 | 36 |
| 54 | 9'90796 | 9'90887 | 9'90978 | 9'91069 | 9'91158 | 9'91248 | 9'91336 | 35 |
| 55 | 8'91336 | 9'91425 | 9'91512 | 9'91599 | 9'91686 | 9'91772 | 9'91857 | 34 |
| 56 | 9'91857 | 9'91942 | 9'92027 | 9'92111 | 9'92194 | 9'92277 | 9'92359 | 33 |
| 57 | 9'92359 | 9'92441 | 9'92522 | 9'92603 | 9'92683 | 9'92763 | 9'92842 | 32 |
| 58 | 9'92842 | 9'92921 | 9'92999 | 9'93077 | 9'93154 | 9'93230 | 9'93307 | 31 |
| 59 | 9'93307 | 9'93382 | 9'93457 | 9'93532 | 9'93606 | 9'93680 | 9'93753 | 30 |
| 60 | 9'93753 | 9'93826 | 9'93898 | 9'93970 | 9'94041 | 9'94112 | 9'94182 | 29 |
| 61 | 9'94182 | 9'94252 | 9'94321 | 9'94390 | 9'94458 | 9'94526 | 9'94593 | 28 |
| 62 | 9'94593 | 9'94660 | 9'94727 | 9'94793 | 9'94858 | 9'94923 | 9'94988 | 27 |
| 63 | 9'94988 | 9'95052 | 9'95116 | 9'95179 | 9'95242 | 9'95304 | 9'95366 | 26 |
| 64 | 9'95366 | 9'95427 | 9'95488 | 9'95549 | 9'95609 | 9'95668 | 9'95728 | 25 |
| 65 | 9'95728 | 9'95786 | 9'95845 | 9'95902 | 9'95960 | 9'96017 | 9'96073 | 24 |
| 66 | 9'96073 | 9'96129 | 9'96185 | 9'96240 | 9'96294 | 9'96349 | 9'96403 | 23 |
| 67 | 9'96403 | 9'96456 | 9'96509 | 9'96562 | 9'96614 | 9'96665 | 9'96717 | 22 |
| 68 | 9'96717 | 9'96769 | 9'96818 | 9'96868 | 9'96917 | 9'96966 | 9'97015 | 21 |
| 69 | 9'97015 | 9'97063 | 9'97111 | 9'97159 | 9'97206 | 9'97252 | 9'97299 | 20 |
| 70 | 4'97299 | 9'97344 | 9'97390 | 9'97435 | 9'97479 | 9'97523 | 9'97567 | 19 |
| 71 | 9'97567 | 9'97610 | 9'97653 | 9'97696 | 9'97738 | 9'97779 | 9'97821 | 18 |
| 72 | 9'97821 | 9'97861 | 9'97902 | 9'97942 | 9'97982 | 9'98021 | 9'98060 | 17 |
| 73 | 9'98060 | 9'98098 | 9'98136 | 9'98174 | 9'98211 | 9'98248 | 9'98284 | 16 |
| 74 | 9'98284 | 9'98320 | 9'98356 | 9'98391 | 9'98426 | 9'98460 | 9'98494 | 15 |
| 75 | 9'98494 | 9'98528 | 9'98561 | 9'98594 | 9'98627 | 9'98659 | 9'98690 | 14 |
| 76 | 9'98690 | 9'98722 | 9'98753 | 9'98783 | 9'98813 | 9'98843 | 9'98872 | 13 |
| 77 | 9'98872 | 9'98901 | 9'98930 | 9'98958 | 9'98986 | 9'99013 | 9'99040 | 12 |
| 78 | 9'99040 | 9'99067 | 9'99093 | 9'99119 | 9'99145 | 9'99170 | 9'99195 | 11 |
| 79 | 9'99195 | 9'99219 | 9'99243 | 9'99267 | 9'99290 | 9'99313 | 9'99335 | 10 |
| 80 | 9'99335 | 9'99357 | 9'99379 | 9'99400 | 9'99421 | 9'99442 | 9'99462 | 9 |
| 81 | 9'99462 | 9'99482 | 9'99501 | 9'99520 | 9'99539 | 9'99557 | 9'99575 | 8 |
| 82 | 9'99575 | 9'99593 | 9'99610 | 9'99627 | 9'99643 | 9'99659 | 9'99675 | 7 |
| 83 | 9'99675 | 9'99690 | 9'99705 | 9'99720 | 9'99734 | 9'99748 | 9'99761 | 6 |
| 84 | 9'99761 | 9'99775 | 9'99787 | 9'99800 | 9'99812 | 9'99823 | 9'99834 | 5 |
| 85 | 9'99834 | 9'99845 | 9'99856 | 9'99866 | 9'99876 | 9'99885 | 9'99894 | 4 |
| 86 | 9'99894 | 9'99903 | 9'99911 | 9'99919 | 9'99926 | 9'99934 | 9'99940 | 3 |
| 87 | 9'99940 | 9'99947 | 9'99953 | 9'99959 | 9'99964 | 9'99969 | 9'99974 | 2 |
| 88 | 9'99974 | 9'99978 | 9'99982 | 9'99985 | 9'99988 | 9'99991 | 9'99993 | 1 |
| 89 | 9'99993 | 9'99995 | 9'99997 | 9'99998 | 9'99999 | 9'99999 | 10'00000 | 0 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cosines |

| Tan- gents | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|---------------|---------|---------|---------|---------|---------|---------|----------|-----------------|
| 0° | - ∞ | 7.46373 | 7.76476 | 7.94086 | 8.06581 | 8.16273 | 8.24192 | 89° |
| 1 | 8.24192 | 8.30888 | 8.36689 | 8.41807 | 8.46385 | 8.50527 | 8.54308 | 88 |
| 2 | 8.54308 | 8.57788 | 8.61009 | 8.64009 | 8.66818 | 8.69453 | 8.71940 | 87 |
| 3 | 8.71940 | 8.74292 | 8.76525 | 8.78649 | 8.80674 | 8.82610 | 8.84464 | 86 |
| 4 | 8.84464 | 8.86243 | 8.87953 | 8.89598 | 8.91185 | 8.92716 | 8.94195 | 85 |
| 5 | 8.94195 | 8.95627 | 8.97013 | 8.98358 | 8.99662 | 9.00930 | 9.02162 | 84 |
| 6 | 9.02162 | 9.03361 | 9.04528 | 9.05666 | 9.06775 | 9.07858 | 9.08914 | 83 |
| 7 | 9.08914 | 9.09947 | 9.10956 | 9.11943 | 9.12909 | 9.13854 | 9.14780 | 82 |
| 8 | 9.14780 | 9.15688 | 9.16577 | 9.17450 | 9.18306 | 9.19146 | 9.19971 | 81 |
| 9 | 9.19971 | 9.20782 | 9.21578 | 9.22361 | 9.23130 | 9.23887 | 9.24632 | 80 |
| 10 | 9.24632 | 9.25365 | 9.26086 | 9.26797 | 9.27496 | 9.28186 | 9.28865 | 79 |
| 11 | 9.28865 | 9.29535 | 9.30195 | 9.30846 | 9.31489 | 9.32122 | 9.32747 | 78 |
| 12 | 9.32747 | 9.33365 | 9.33974 | 9.34576 | 9.35170 | 9.35757 | 9.36336 | 77 |
| 13 | 9.36336 | 9.36909 | 9.37476 | 9.38035 | 9.38589 | 9.39136 | 9.39677 | 76 |
| 14 | 9.39677 | 9.40212 | 9.40742 | 9.41266 | 9.41784 | 9.42297 | 9.42805 | 75 |
| 15 | 9.42805 | 9.43308 | 9.43806 | 9.44299 | 9.44787 | 9.45271 | 9.45750 | 74 |
| 16 | 9.45750 | 9.46224 | 9.46694 | 9.47160 | 9.47622 | 9.48080 | 9.48534 | 73 |
| 17 | 9.48534 | 9.48984 | 9.49430 | 9.49872 | 9.50311 | 9.50746 | 9.51178 | 72 |
| 18 | 9.51178 | 9.51606 | 9.52031 | 9.52452 | 9.52870 | 9.53285 | 9.53697 | 71 |
| 19 | 9.53697 | 9.54106 | 9.54512 | 9.54915 | 9.55315 | 9.55712 | 9.56107 | 70 |
| 20 | 9.56107 | 9.56458 | 9.56887 | 9.57274 | 9.57658 | 9.58039 | 9.58418 | 69 |
| 21 | 9.58418 | 9.58794 | 9.59168 | 9.59540 | 9.59909 | 9.60276 | 9.60641 | 68 |
| 22 | 9.60641 | 9.61004 | 9.61364 | 9.61722 | 9.62079 | 9.62433 | 9.62785 | 67 |
| 23 | 9.62785 | 9.63135 | 9.63484 | 9.63830 | 9.64175 | 9.64517 | 9.64858 | 66 |
| 24 | 9.64858 | 9.65197 | 9.65535 | 9.65870 | 9.66204 | 9.66537 | 9.66867 | 65 |
| 25 | 9.66867 | 9.67196 | 9.67524 | 9.67850 | 9.68174 | 9.68497 | 9.68818 | 64 |
| 26 | 9.68818 | 9.69138 | 9.69457 | 9.69774 | 9.70089 | 9.70404 | 9.70717 | 63 |
| 27 | 9.70717 | 9.71028 | 9.71339 | 9.71648 | 9.71955 | 9.72262 | 9.72567 | 62 |
| 28 | 9.72567 | 9.72872 | 9.73175 | 9.73476 | 9.73777 | 9.74077 | 9.74375 | 61 |
| 29 | 9.74375 | 9.74673 | 9.74969 | 9.75264 | 9.75558 | 9.75852 | 9.76144 | 60 |
| 30 | 9.76144 | 9.76435 | 9.76726 | 9.77015 | 9.77303 | 9.77591 | 9.77877 | 59 |
| 31 | 9.77877 | 9.78163 | 9.78448 | 9.78732 | 9.79015 | 9.79297 | 9.79579 | 58 |
| 32 | 9.79579 | 9.79860 | 9.80140 | 9.80419 | 9.80697 | 9.80975 | 9.81252 | 57 |
| 33 | 9.81252 | 9.81528 | 9.81803 | 9.82078 | 9.82352 | 9.82626 | 9.82899 | 56 |
| 34 | 9.82899 | 9.83171 | 9.83442 | 9.83713 | 9.83984 | 9.84254 | 9.84523 | 55 |
| 35 | 9.84523 | 9.84791 | 9.85059 | 9.85327 | 9.85594 | 9.85860 | 9.86126 | 54 |
| 36 | 9.86126 | 9.86392 | 9.86656 | 9.86921 | 9.87185 | 9.87448 | 9.87711 | 53 |
| 37 | 9.87711 | 9.87974 | 9.88236 | 9.88498 | 9.88759 | 9.89020 | 9.89281 | 52 |
| 38 | 9.89281 | 9.89541 | 9.89801 | 9.90061 | 9.90320 | 9.90578 | 9.90837 | 51 |
| 39 | 9.90837 | 9.91095 | 9.91353 | 9.91610 | 9.91868 | 9.92125 | 9.92381 | 50 |
| 40 | 9.92381 | 9.92638 | 9.92894 | 9.93150 | 9.93406 | 9.93661 | 9.93916 | 49 |
| 41 | 9.93916 | 9.94171 | 9.94426 | 9.94681 | 9.94935 | 9.95189 | 9.95444 | 48 |
| 42 | 9.95444 | 9.95698 | 9.95952 | 9.96205 | 9.96459 | 9.96712 | 9.96966 | 47 |
| 43 | 9.96966 | 9.97219 | 9.97472 | 9.97725 | 9.97978 | 9.98231 | 9.98484 | 46 |
| 44 | 9.98484 | 9.98737 | 9.98989 | 9.99242 | 9.99495 | 9.99747 | 10.00000 | 45 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cotan- gents |

| Tan- gents | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|---------------|----------|----------|----------|----------|----------|----------|----------|-----------------|
| 45° | 10°00000 | 10°00253 | 10°00505 | 10°00758 | 10°01011 | 10°01263 | 10°01516 | 44° |
| 46 | 10°01516 | 10°01769 | 10°02022 | 10°02275 | 10°02528 | 10°02781 | 10°03034 | 43 |
| 47 | 10°03034 | 10°03288 | 10°03541 | 10°03795 | 10°04048 | 10°04302 | 10°04556 | 42 |
| 48 | 10°04556 | 10°04810 | 10°05065 | 10°05319 | 10°05574 | 10°05829 | 10°06084 | 41 |
| 49 | 10°06084 | 10°06339 | 10°06594 | 10°06850 | 10°07106 | 10°07362 | 10°07619 | 40 |
| 50 | 10°07619 | 10°07875 | 10°08132 | 10°08390 | 10°08647 | 10°08905 | 10°09163 | 39 |
| 51 | 10°09163 | 10°09422 | 10°09680 | 10°09939 | 10°10199 | 10°10459 | 10°10719 | 38 |
| 52 | 10°10719 | 10°10980 | 10°11241 | 10°11502 | 10°11764 | 10°12026 | 10°12289 | 37 |
| 53 | 10°12289 | 10°12552 | 10°12815 | 10°13079 | 10°13344 | 10°13608 | 10°13874 | 36 |
| 54 | 10°13874 | 10°14140 | 10°14406 | 10°14673 | 10°14941 | 10°15209 | 10°15477 | 35 |
| 55 | 10°15477 | 10°15746 | 10°16016 | 10°16287 | 10°16558 | 10°16829 | 10°17101 | 34 |
| 56 | 10°17101 | 10°17374 | 10°17648 | 10°17922 | 10°18197 | 10°18472 | 10°18748 | 33 |
| 57 | 10°18748 | 10°19025 | 10°19303 | 10°19581 | 10°19860 | 10°20140 | 10°20421 | 32 |
| 58 | 10°20421 | 10°20703 | 10°20985 | 10°21268 | 10°21552 | 10°21837 | 10°22123 | 31 |
| 59 | 10°22123 | 10°22409 | 10°22697 | 10°22985 | 10°23275 | 10°23565 | 10°23856 | 30 |
| 60 | 10°23856 | 10°24148 | 10°24442 | 10°24736 | 10°25031 | 10°25327 | 10°25625 | 29 |
| 61 | 10°25625 | 10°25923 | 10°26223 | 10°26524 | 10°26825 | 10°27128 | 10°27433 | 28 |
| 62 | 10°27433 | 10°27738 | 10°28045 | 10°28352 | 10°28661 | 10°28972 | 10°29283 | 27 |
| 63 | 10°29283 | 10°29596 | 10°29911 | 10°30226 | 10°30543 | 10°30862 | 10°31182 | 26 |
| 64 | 10°31182 | 10°31503 | 10°31826 | 10°32150 | 10°32476 | 10°32804 | 10°33133 | 25 |
| 65 | 10°33133 | 10°33463 | 10°33796 | 10°34130 | 10°34465 | 10°34803 | 10°35142 | 24 |
| 66 | 10°35142 | 10°35483 | 10°35825 | 10°36170 | 10°36516 | 10°36865 | 10°37215 | 23 |
| 67 | 10°37215 | 10°37567 | 10°37921 | 10°38278 | 10°38636 | 10°38996 | 10°39359 | 22 |
| 68 | 10°39359 | 10°39724 | 10°40091 | 10°40460 | 10°40832 | 10°41206 | 10°41582 | 21 |
| 69 | 10°41582 | 10°41961 | 10°42342 | 10°42726 | 10°43113 | 10°43502 | 10°43893 | 20 |
| 70 | 10°43893 | 10°44288 | 10°44685 | 10°45085 | 10°45488 | 10°45894 | 10°46303 | 19 |
| 71 | 10°46303 | 10°46715 | 10°47130 | 10°47548 | 10°47969 | 10°48394 | 10°48822 | 18 |
| 72 | 10°48822 | 10°49254 | 10°49689 | 10°50128 | 10°50570 | 10°51016 | 10°51466 | 17 |
| 73 | 10°51466 | 10°51920 | 10°52378 | 10°52840 | 10°53306 | 10°53776 | 10°54250 | 16 |
| 74 | 10°54250 | 10°54729 | 10°55213 | 10°55701 | 10°56194 | 10°56692 | 10°57195 | 15 |
| 75 | 10°57195 | 10°57703 | 10°58216 | 10°58734 | 10°59258 | 10°59788 | 10°60323 | 14 |
| 76 | 10°60323 | 10°60864 | 10°61411 | 10°61965 | 10°62524 | 10°63091 | 10°63664 | 13 |
| 77 | 10°63664 | 10°64234 | 10°64830 | 10°65424 | 10°66026 | 10°66635 | 10°67253 | 12 |
| 78 | 10°67253 | 10°67878 | 10°68511 | 10°69154 | 10°69805 | 10°70465 | 10°71135 | 11 |
| 79 | 10°71135 | 10°71814 | 10°72504 | 10°73203 | 10°73914 | 10°74635 | 10°75368 | 10 |
| 80 | 10°75368 | 10°76113 | 10°76870 | 10°77639 | 10°78422 | 10°79218 | 10°80029 | 9 |
| 81 | 10°80029 | 10°80854 | 10°81694 | 10°82550 | 10°83423 | 10°84312 | 10°85220 | 8 |
| 82 | 10°85220 | 10°86146 | 10°87091 | 10°88057 | 10°89044 | 10°90053 | 10°91086 | 7 |
| 83 | 10°91086 | 10°92142 | 10°93225 | 10°94334 | 10°95472 | 10°96639 | 10°97838 | 6 |
| 84 | 10°97838 | 10°99070 | 11°00338 | 11°01642 | 11°02987 | 11°04373 | 11°05805 | 5 |
| 85 | 11°05805 | 11°07284 | 11°08815 | 11°10402 | 11°12047 | 11°13757 | 11°15536 | 4 |
| 86 | 11°15536 | 11°17390 | 11°19326 | 11°21351 | 11°23475 | 11°25708 | 11°28060 | 3 |
| 87 | 11°28060 | 11°30547 | 11°33184 | 11°35991 | 11°38991 | 11°42212 | 11°45692 | 2 |
| 88 | 11°45692 | 11°49473 | 11°53615 | 11°58193 | 11°63311 | 11°69112 | 11°75808 | 1 |
| 89 | 11°75808 | 11°83727 | 11°93419 | 12°05914 | 12°23524 | 12°53627 | ∞ | 0 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cotan- gents |

| Secants | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|---------|----------|----------|----------|----------|----------|----------|----------|------------|
| 0° | 10°00000 | 10°00001 | 10°00001 | 10°00002 | 10°00003 | 10°00005 | 10°00007 | 89° |
| 1 | 10°00007 | 10°00009 | 10°00012 | 10°00015 | 10°00018 | 10°00022 | 10°00026 | 88 |
| 2 | 10°00026 | 10°00031 | 10°00036 | 10°00041 | 10°00047 | 10°00053 | 10°00060 | 87 |
| 3 | 10°00060 | 10°00066 | 10°00074 | 10°00081 | 10°00089 | 10°00097 | 10°00106 | 86 |
| 4 | 10°00106 | 10°00115 | 10°00124 | 10°00134 | 10°00144 | 10°00155 | 10°00166 | 85 |
| 5 | 10°00166 | 10°00177 | 10°00188 | 10°00200 | 10°00213 | 10°00225 | 10°00239 | 84 |
| 6 | 10°00239 | 10°00252 | 10°00266 | 10°00280 | 10°00295 | 10°00310 | 10°00325 | 83 |
| 7 | 10°00325 | 10°00341 | 10°00357 | 10°00373 | 10°00390 | 10°00407 | 10°00425 | 82 |
| 8 | 10°00425 | 10°00443 | 10°00461 | 10°00480 | 10°00499 | 10°00518 | 10°00538 | 81 |
| 9 | 10°00538 | 10°00558 | 10°00579 | 10°00600 | 10°00621 | 10°00643 | 10°00665 | 80 |
| 10 | 10°00665 | 10°00687 | 10°00710 | 10°00733 | 10°00757 | 10°00781 | 10°00805 | 79 |
| 11 | 10°00805 | 10°00830 | 10°00855 | 10°00881 | 10°00907 | 10°00933 | 10°00960 | 78 |
| 12 | 10°00960 | 10°00987 | 10°01014 | 10°01041 | 10°01070 | 10°01099 | 10°01128 | 77 |
| 13 | 10°01128 | 10°01157 | 10°01187 | 10°01217 | 10°01247 | 10°01278 | 10°01310 | 76 |
| 14 | 10°01310 | 10°01341 | 10°01373 | 10°01406 | 10°01439 | 10°01472 | 10°01506 | 75 |
| 15 | 10°01506 | 10°01540 | 10°01574 | 10°01609 | 10°01644 | 10°01680 | 10°01716 | 74 |
| 16 | 10°01716 | 10°01752 | 10°01789 | 10°01826 | 10°01864 | 10°01902 | 10°01940 | 73 |
| 17 | 10°01940 | 10°01979 | 10°02018 | 10°02058 | 10°02098 | 10°02139 | 10°02179 | 72 |
| 18 | 10°02179 | 10°02221 | 10°02262 | 10°02304 | 10°02347 | 10°02390 | 10°02433 | 71 |
| 19 | 10°02433 | 10°02477 | 10°02521 | 10°02565 | 10°02610 | 10°02656 | 10°02701 | 70 |
| 20 | 10°02701 | 10°02748 | 10°02794 | 10°02841 | 10°02889 | 10°02937 | 10°02985 | 69 |
| 21 | 10°02985 | 10°03034 | 10°03083 | 10°03132 | 10°03182 | 10°03233 | 10°03283 | 68 |
| 22 | 10°03283 | 10°03335 | 10°03386 | 10°03438 | 10°03491 | 10°03544 | 10°03597 | 67 |
| 23 | 10°03597 | 10°03651 | 10°03706 | 10°03760 | 10°03815 | 10°03871 | 10°03927 | 66 |
| 24 | 10°03927 | 10°03983 | 10°04040 | 10°04098 | 10°04156 | 10°04214 | 10°04272 | 65 |
| 25 | 10°04272 | 10°04332 | 10°04391 | 10°04451 | 10°04511 | 10°04573 | 10°04634 | 64 |
| 26 | 10°04634 | 10°04696 | 10°04758 | 10°04821 | 10°04884 | 10°04948 | 10°05012 | 63 |
| 27 | 10°05012 | 10°05077 | 10°05142 | 10°05207 | 10°05273 | 10°05340 | 10°05407 | 62 |
| 28 | 10°05407 | 10°05474 | 10°05542 | 10°05610 | 10°05679 | 10°05748 | 10°05818 | 61 |
| 29 | 10°05818 | 10°05888 | 10°05959 | 10°06030 | 10°06102 | 10°06174 | 10°06247 | 60 |
| 30 | 10°06247 | 10°06320 | 10°06394 | 10°06468 | 10°06543 | 10°06618 | 10°06693 | 59 |
| 31 | 10°06693 | 10°06770 | 10°06846 | 10°06923 | 10°07001 | 10°07079 | 10°07158 | 58 |
| 32 | 10°07158 | 10°07237 | 10°07317 | 10°07397 | 10°07478 | 10°07559 | 10°07641 | 57 |
| 33 | 10°07641 | 10°07723 | 10°07806 | 10°07889 | 10°07973 | 10°08058 | 10°08143 | 56 |
| 34 | 10°08143 | 10°08228 | 10°08314 | 10°08401 | 10°08488 | 10°08575 | 10°08664 | 55 |
| 35 | 10°08664 | 10°08752 | 10°08842 | 10°08931 | 10°09022 | 10°09113 | 10°09204 | 54 |
| 36 | 10°09204 | 10°09296 | 10°09389 | 10°09482 | 10°09576 | 10°09670 | 10°09765 | 53 |
| 37 | 10°09765 | 10°09861 | 10°09957 | 10°10053 | 10°10151 | 10°10248 | 10°10347 | 52 |
| 38 | 10°10347 | 10°10446 | 10°10545 | 10°10646 | 10°10746 | 10°10848 | 10°10950 | 51 |
| 39 | 10°10950 | 10°11052 | 10°11156 | 10°11259 | 10°11364 | 10°11469 | 10°11575 | 50 |
| 40 | 10°11575 | 10°11681 | 10°11788 | 10°11895 | 10°12004 | 10°12113 | 10°12222 | 49 |
| 41 | 10°12222 | 10°12332 | 10°12443 | 10°12554 | 10°12666 | 10°12779 | 10°12893 | 48 |
| 42 | 10°12893 | 10°13007 | 10°13121 | 10°13237 | 10°13353 | 10°13470 | 10°13587 | 47 |
| 43 | 10°13587 | 10°13705 | 10°13824 | 10°13944 | 10°14064 | 10°14185 | 10°14307 | 46 |
| 44 | 10°14307 | 10°14429 | 10°14552 | 10°14676 | 10°14800 | 10°14926 | 10°15052 | 45 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Co-secants |

| Se- cants | 0' | 10' | 20' | 30' | 40' | 50' | 60' | |
|--------------|----------|----------|----------|----------|----------|----------|----------|----------------|
| 45° | 10°15052 | 10°15178 | 10°15306 | 10°15434 | 10°15563 | 10°15692 | 10°15823 | 44° |
| 46 | 10°15823 | 10°15954 | 10°16086 | 10°16219 | 10°16352 | 10°16487 | 10°16622 | 43 |
| 47 | 10°16623 | 10°16758 | 10°16894 | 10°17032 | 10°17170 | 10°17309 | 10°17449 | 42 |
| 48 | 10°17449 | 10°17590 | 10°17731 | 10°17874 | 10°18017 | 10°18161 | 10°18306 | 41 |
| 49 | 10°18306 | 10°18451 | 10°18598 | 10°18746 | 10°18894 | 10°19043 | 10°19193 | 40 |
| 50 | 10°19193 | 10°19344 | 10°19496 | 10°19649 | 10°19803 | 10°19957 | 10°20113 | 39 |
| 51 | 10°20113 | 10°20269 | 10°20427 | 10°20585 | 10°20744 | 10°20905 | 10°21066 | 38 |
| 52 | 10°21066 | 10°21228 | 10°21391 | 10°21555 | 10°21720 | 10°21887 | 10°22054 | 37 |
| 53 | 10°22054 | 10°22222 | 10°22391 | 10°22561 | 10°22732 | 10°22905 | 10°23078 | 36 |
| 54 | 10°23078 | 10°23253 | 10°23428 | 10°23605 | 10°23782 | 10°23961 | 10°24141 | 35 |
| 55 | 10°24141 | 10°24322 | 10°24504 | 10°24687 | 10°24872 | 10°25057 | 10°25243 | 34 |
| 56 | 10°25243 | 10°25432 | 10°25621 | 10°25811 | 10°26003 | 10°26195 | 10°26389 | 33 |
| 57 | 10°26389 | 10°26584 | 10°26781 | 10°26978 | 10°27177 | 10°27378 | 10°27579 | 32 |
| 58 | 10°27579 | 10°27782 | 10°27986 | 10°28191 | 10°28398 | 10°28607 | 10°28816 | 31 |
| 59 | 10°28816 | 10°29027 | 10°29239 | 10°29453 | 10°29668 | 10°29885 | 10°30103 | 30 |
| 60 | 10°30103 | 10°30323 | 10°30544 | 10°30766 | 10°30990 | 10°31216 | 10°31443 | 29 |
| 61 | 10°31443 | 10°31672 | 10°31902 | 10°32134 | 10°32367 | 10°32602 | 10°32839 | 28 |
| 62 | 10°32839 | 10°33078 | 10°33318 | 10°33559 | 10°33803 | 10°34048 | 10°34295 | 27 |
| 63 | 10°34295 | 10°34544 | 10°34795 | 10°35047 | 10°35302 | 10°35557 | 10°35816 | 26 |
| 64 | 10°35816 | 10°36076 | 10°36338 | 10°36602 | 10°36867 | 10°37135 | 10°37405 | 25 |
| 65 | 10°37405 | 10°37677 | 10°37951 | 10°38227 | 10°38506 | 10°38786 | 10°39069 | 24 |
| 66 | 10°39069 | 10°39354 | 10°39641 | 10°39930 | 10°40222 | 10°40516 | 10°40812 | 23 |
| 67 | 10°40812 | 10°41111 | 10°41412 | 10°41716 | 10°42022 | 10°42331 | 10°42642 | 22 |
| 68 | 10°42642 | 10°42956 | 10°43273 | 10°43592 | 10°43915 | 10°44239 | 10°44567 | 21 |
| 69 | 10°44567 | 10°44898 | 10°45231 | 10°45567 | 10°45907 | 10°46249 | 10°46595 | 20 |
| 70 | 10°46595 | 10°46944 | 10°47295 | 10°47650 | 10°48009 | 10°48371 | 10°48736 | 19 |
| 71 | 10°48736 | 10°49104 | 10°49477 | 10°49852 | 10°50232 | 10°50615 | 10°51002 | 18 |
| 72 | 10°51002 | 10°51393 | 10°51787 | 10°52186 | 10°52589 | 10°52995 | 10°53406 | 17 |
| 73 | 10°53406 | 10°53822 | 10°54242 | 10°54666 | 10°55095 | 10°55528 | 10°55966 | 16 |
| 74 | 10°55966 | 10°56409 | 10°56857 | 10°57310 | 10°57768 | 10°58232 | 10°58700 | 15 |
| 75 | 10°58700 | 10°59175 | 10°59654 | 10°60140 | 10°60631 | 10°61129 | 10°61632 | 14 |
| 76 | 10°61632 | 10°62142 | 10°62659 | 10°63181 | 10°63711 | 10°64248 | 10°64791 | 13 |
| 77 | 10°64791 | 10°65342 | 10°65900 | 10°66466 | 10°67040 | 10°67622 | 10°68212 | 12 |
| 78 | 10°68212 | 10°68811 | 10°69418 | 10°70034 | 10°70660 | 10°71295 | 10°71940 | 11 |
| 79 | 10°71940 | 10°72595 | 10°73261 | 10°73937 | 10°74624 | 10°75323 | 10°76033 | 10 |
| 80 | 10°76033 | 10°76756 | 10°77491 | 10°78239 | 10°79001 | 10°79777 | 10°80567 | 9 |
| 81 | 10°80567 | 10°81372 | 10°82193 | 10°83030 | 10°83884 | 10°84755 | 10°85644 | 8 |
| 82 | 10°85644 | 10°86553 | 10°87481 | 10°88430 | 10°89401 | 10°90394 | 10°91411 | 7 |
| 83 | 10°91411 | 10°92452 | 10°93519 | 10°94614 | 10°95738 | 10°96891 | 10°98077 | 6 |
| 84 | 10°98077 | 10°99296 | 11°00550 | 11°01843 | 11°03175 | 11°04550 | 11°05970 | 5 |
| 85 | 11°05970 | 11°07439 | 11°08960 | 11°10536 | 11°12171 | 11°13872 | 11°15642 | 4 |
| 86 | 11°15642 | 11°17487 | 11°19415 | 11°21432 | 11°23549 | 11°25774 | 11°28120 | 3 |
| 87 | 11°28120 | 11°30600 | 11°33231 | 11°36032 | 11°39027 | 11°42243 | 11°45718 | 2 |
| 88 | 11°45718 | 11°49496 | 11°53634 | 11°58208 | 11°63322 | 11°69121 | 11°75814 | 1 |
| 89 | 11°75814 | 11°83732 | 11°93422 | 12°05916 | 12°23525 | 12°53627 | ∞ | 0 |
| | 60' | 50' | 40' | 30' | 20' | 10' | 0' | Cose- cants |

§ 16. Discrepancies due to irregularity and insensibility.

In § 7, when discussing the use of Tables of Natural Sines, Cosines, etc., it was stated that roughly four-, five-, and seven-figure Tables could be trusted to give results correct to four, five and seven significant figures.

It is most important, in any calculation in which tables are used, to know to what extent they can be trusted to give correct results.

Discrepancies may arise in interpolation from one of two causes,

- (i) **Irregularity**; i.e., there is a rapid change in the magnitude of consecutive differences.
- (ii) **Insensibility**; i.e., the differences are too small.

If a graph of a particular set of tables is drawn, irregularity is shown by a rapid change of slope of the corresponding part of the curve; whilst insensibility is shown by the curve becoming parallel to the x -axis.

Thus in a table of natural sines we find

| I. | Angle | 0° | 0° 10' | 0° 20' | 0° 30' | 0° 40' | 0° 50' | 1° 0' |
|----|-------|--------|--------|--------|--------|--------|--------|--------|
| | Sine | ·00000 | ·00291 | ·00582 | ·00873 | ·01164 | ·01454 | ·01745 |
| | Diff. | | 291 | 291 | 291 | 291 | 290 | 290 |

| II. | Angle | 89° 0' | 89° 10' | 89° 20' | 89° 30' | 89° 40' | 89° 50' | 90° |
|-----|-------|--------|---------|---------|---------|---------|---------|---------|
| | Sine | ·99985 | ·99989 | ·99993 | ·99996 | ·99998 | ·99999 | 1·00000 |
| | Diff. | | 4 | 4 | 3 | 2 | 1 | 1 |

Thus in the early part of the tables, the differences are regular and fairly sensible: a difference of 290 in the sine corresponds to a difference of $10'$ in the angle, and thus the tables can be trusted to give results true to about $2''$ or $3''$ of angle.

On the other hand when the angle is nearly a right angle, the differences are not only rather irregular but very small; as a difference of 4 in the sine corresponds to a difference of $10'$ in the angle, the tables cannot be trusted to give results true to even $2'$ of angle, and are thus practically useless for this range. However, this difficulty can as a rule be avoided by using a different formula, involving one of the other trigonometrical ratios; for irregularity and insensibility occur for different ranges in different tables.

Thus for cosines the values for angles nearly 90° are regular and sensible, whilst for very small angles there is some irregularity, and great insensibility.

1. Examine the tables of tangents, cotangents, secants and cosecants, for irregularity and insensibility.

2. State for what ranges, for each trigonometrical ratio, the tables can be relied upon if we wish to secure results correct to one-tenth of a minute.

3. Find, from the five-figure tables on pp. 36—41, the values of

- (1) $\sin 2^\circ 13' 27''$, $\sin 6^\circ 27' 15''$, $\cos 79^\circ 12' 20''$, $\cos 87^\circ 34' 49''$;
- (2) $\tan 60^\circ 21' 45''$, $\tan 79^\circ 31' 2''$, $\cot 43^\circ 2' 40''$, $\cot 4^\circ 37' 37''$;
- (3) $\sec 54^\circ 54' 54''$, $\sec 82^\circ 29' 42''$, $\operatorname{cosec} 31^\circ 42' 29''$, $\operatorname{cosec} 6^\circ 52' 16''$;

and compare the results obtained with the seven-figure results given in the answers.

The tables of logarithms to the base 10, or **common logarithms** as they are generally called, given on pp. 88, 89, shew that there is some irregularity for numbers between 1·0000 and 1·1000.

Thus

$$\begin{array}{rcl}
 \log 1\cdot0000 = 0\cdot00000 & \left. \vphantom{\log 1\cdot0000} \right\} & \text{diff.} = 432 \\
 \log 1\cdot0100 = 0\cdot00432 & \left. \vphantom{\log 1\cdot0100} \right\} & \text{diff.} = 428 \\
 \log 1\cdot0200 = 0\cdot00860 & \left. \vphantom{\log 1\cdot0200} \right\} & \\
 \dots\dots\dots & & \\
 \log 1\cdot0800 = 0\cdot03342 & \left. \vphantom{\log 1\cdot0800} \right\} & \text{diff.} = 401 \\
 \log 1\cdot0900 = 0\cdot03743 & \left. \vphantom{\log 1\cdot0900} \right\} & \text{diff.} = 396. \\
 \log 1\cdot1000 = 0\cdot04139 & \left. \vphantom{\log 1\cdot1000} \right\} &
 \end{array}$$

Except for this irregularity, the differences are quite large enough to give by interpolation two more significant figures instead of the zeros for a number between any two of the above.

For example, if it is required to find $\log 1\cdot0137$, we get by interpolation

$$\begin{array}{r}
 \log 1\cdot0137 = 0\cdot00432 \\
 + \quad 158 \\
 = 0\cdot00590
 \end{array}$$

whilst the correct value is 0·00591.

Hence, as it is evident that the irregularity becomes greater, the nearer the logarithm approaches 1·0000, the tables have been extended for differences of only 0·0010 between 1·0000 and 1·1000, leaving only one more significant figure to be found by interpolation.

4. Find, from the five-figure tables on pp. 88, 89, the values of

$$\begin{array}{l}
 \log 100\cdot57, \log 1000\cdot3, \log 0\cdot102176, \log 103\cdot752, \\
 \log 0\cdot0108276, \log 1\cdot00002, \log 1098\cdot733,
 \end{array}$$

and compare the results with the seven-figure results in the answers.

At the other end of the table the differences are very regular, but are considerably smaller.

Thus

| | |
|-------------------------|--------------|
| $\log 8.9000 = 0.94939$ | } diff. = 49 |
| $\log 8.9100 = 0.94988$ | |
| $\log 8.9200 = 0.95036$ | } diff. = 48 |
| $\log 8.9300 = 0.95085$ | |
| | |
| $\log 8.9700 = 0.95279$ | } diff. = 49 |
| $\log 8.9800 = 0.95328$ | |
| $\log 8.9900 = 0.95376$ | } diff. = 48 |
| $\log 9.0000 = 0.95424$ | |

Hence, interpolation will give two more significant figures correctly; but, since the difference for 0.0100 is only 48, the difference for 0.0001 is roughly only $\frac{1}{2}$. Therefore we shall have two, or perhaps three in some cases, consecutive numbers with the same logarithms; so that although the logarithms of numbers with five significant figures can be found correctly by interpolation, yet if the logarithm is given, and the corresponding number has to be found by interpolation, there may be a mistake of 1 (or in rare cases, 2) in the fifth significant figure. But this is only an error of at most 2 in 80,000, i.e. considerably less than 1 in 10,000: thus the table is *uniformly* correct to .01 %.

5. Find, from the five-figure tables on pp. 88, 89, the values of

(1) $\log 899.72$, $\log 892.6$, $\log 0.723457$, $\log 9.82679$;

(2) *antilog 2.98764, antilog 3.95045, antilog 1.95399;

comparing the results with the seven-figure results in the answers, and in (2) working out the error per cent.

* "Antilog" is the inverse notation for "the number whose logarithm is."

Tables of Tabular Logarithms of the Trigonometrical tables are similarly subject to irregularity and insensibility.

Too much reliance should not be placed on results obtained by interpolation,

(1) owing to *irregularity*, on the values of $L \sin$, $L \cos$, $L \tan$ of angles

between 0° and 5° , and 85° and 90° :

(2) owing to *insensibility*, on the values of \sin^{-1} , \cos^{-1} , obtained from

$L \sin$ of angles between 85° and 90° ,

$L \cos$ of angles between 0° and 5° .

For the generality of angles between 5° and 85° , the differences for $10'$ of angle as given in the tables lie between 20 and 600: hence the tables may be trusted to give results correct to varying amounts from $\frac{1}{2}''$ to $1''$. In working problems the student should always notice and record the possible error in his results, remembering that insensibility only affects them one way, whilst irregularity affects them both ways. Thus $L \cos 6^\circ 42' 37''$ can be found correct to five places, from the tables on pp. 90, 91, to be 9.99701: whereas if it is given that $L \cos x = 9.99701$, x may lie anywhere between $6^\circ 42' 36''$ and $6^\circ 13' 16''$.

When the possible error recorded is due to insensibility recourse must be had to tables calculated to a larger number of significant figures, if the possible error exceeds what is allowable on account of the nature of the problem: if the error is due to irregularity, tables calculated for smaller intervals must be used.

§ 17. Formulae adapted to Logarithms.

It has been shewn in § 15, that formulae which mainly consist of products, quotients, roots, and powers, are specially adapted to logarithmic computation. Of the formulae obtained in §§ 12, 13, viz.:

$$(i) \quad \Delta = \frac{1}{2} bc \sin A,$$

$$(ii) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$(iii) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

only (i) and (ii) are specially adapted to logarithms.

By means of (i) the Area of a triangle can be calculated when any *two sides and the included angle* are given.

By combining (i) and (ii) a formula can be obtained which gives the area when *one side and two (i.e. three) angles* are given.

Thus,

$$\begin{aligned} \Delta &= \frac{1}{2} bc \sin A \\ &= \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C} \dots\dots\dots(iv). \end{aligned}$$

If *two sides and an angle* (not the included angle) are given—say *A, a, b*—it is necessary to find the value or values of *B* by formula (ii), and thence the value or values of *C* and finally use formula (i).

If *three sides* are given, a formula can be found by eliminating A from formulae (ii) and (iii), thus:—

$$2bc \cos A = b^2 + c^2 - a^2,$$

$$2bc \sin A = 4\Delta.$$

Squaring and adding, since $\cos^2 A + \sin^2 A = 1$,

$$\therefore 4b^2c^2 = (b^2 + c^2 - a^2)^2 + 16\Delta^2,$$

$$\begin{aligned}\therefore 16\Delta^2 &= (b^2 + c^2 - a^2)^2 - (2bc)^2 \\ &= (a + b + c)(b + c - a)(c + a - b)(a + b - c).\end{aligned}$$

Let $2s = (a + b + c)$, so that s is the *Semiperimeter* of the triangle, then

$$\Delta = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)} \quad \dots\dots(v).$$

The expressions s , $(s - a)$, $(s - b)$, $(s - c)$ are very useful and important. They are the lengths of certain lines connected with the triangle and the four circles, which touch the three sides, the inscribed circle and the three escribed circles.

69. Let ABC be a triangle, DEF the inscribed circle, let AF , AE be x units, BF , BD , y units, CD , CE , z units of length respectively.

Shew that

$$x + y + z = s,$$

and hence $x = s - a,$

$$y = s - b,$$

$$z = s - c.$$



FIG. 49.

70. Let ABC be a triangle, D_1, E_1, F_1 the escribed circle which touches BC and the other two sides produced.

Shew that

$$AF_1 = AE_1 = s,$$

$$BF_1 = BD_1 = s - a,$$

$$CE_1 = CD_1 = s - b.$$

71. Let ABC be a triangle, I the centre of the inscribed circle DEF , r the radius of the circle.

Shew that

$$\Delta = \frac{1}{2} a \cdot r + \frac{1}{2} b \cdot r + \frac{1}{2} c \cdot r,$$

and hence

$$r = \frac{\Delta}{s}.$$

72. Let ABC be a triangle, I_1 the centre of the escribed circle $D_1E_1F_1$, which touches BC and the other two sides produced, r_1 the radius of the circle.

Shew that

$$\Delta = \frac{1}{2} b \cdot r_1 + \frac{1}{2} c \cdot r_1 - \frac{1}{2} a r_1,$$

and hence

$$r_1 = -\frac{\Delta}{s - a}.$$

Similarly

$$r_2 = \frac{\Delta}{s - b}, \quad r_3 = \frac{\Delta}{s - c}.$$



FIG. 50.

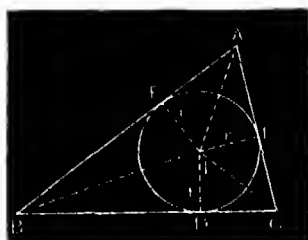


FIG. 51.

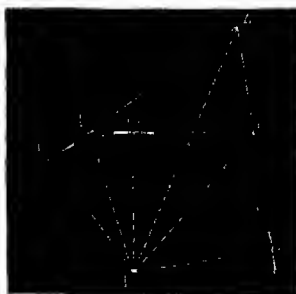


FIG. 52.

73. Let ABC be a triangle, I, r , the centre and radius of the inscribed circle DEF , I_1, r_1 the centre and radius of the escribed circle $D_1E_1F_1$ which touches BC and the other two sides produced. Join I_1D_1, I_1B, ID, IB .

Shew that

$$\angle BI_1D_1 = \angle IBD = \frac{B}{2},$$

and hence

$$\frac{BD_1}{ID_1} = \tan \frac{B}{2} = \frac{ID}{BD},$$

$$\therefore r \cdot r_1 = (s-b)(s-c).$$

Similarly

$$r \cdot r_2 = (s-c)(s-a),$$

$$r \cdot r_3 = (s-a)(s-b).$$

74. Let ABC be a triangle, I, I_1 the centres of the inscribed circle and the escribed circle which touches BC and the other two sides produced.

Shew that $\triangle AIC, ABI_1$ are equiangular and hence, making use of formula (ii), that

$$\frac{AI}{AC} = \frac{AB}{AI_1}.$$

$$\therefore AI \cdot AI_1 = b \cdot c.$$

Similarly shew that

$$BI \cdot BI_2 = c \cdot a,$$

$$CI \cdot CI_3 = a \cdot b.$$

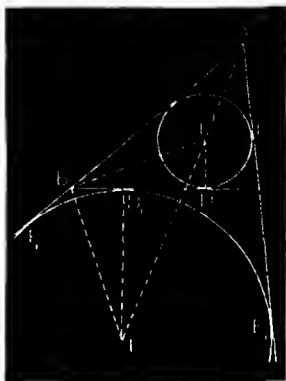


FIG. 53.



FIG. 54.

75. Let ABC be a triangle, I the centre of the inscribed circle DEF , r the radius of this circle.

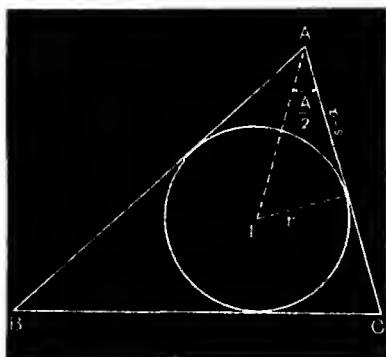


FIG. 55

Shew that AI bisects the angle BAC , and hence

$$\begin{aligned}\tan \frac{A}{2} &= \frac{r}{s-a} \\ &= \frac{\Delta}{s(s-a)} \dots\dots\dots \text{from Expt. 71.} \\ &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \dots\dots (\text{see p. 102, v.})\end{aligned}$$

Similarly

$$\begin{aligned}\tan \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \\ \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \dots\dots\dots (\text{vi}).\end{aligned}$$

These formulae are adapted to logarithmic computation, and are those generally used for the solution of triangles of which *the three sides* are given.

Formulae (v) and (vi), together with corresponding expressions for the sines and cosines of the half-angles can be obtained from Expts. 73, 74.

76. Let ABC be a triangle, I, I_1 the centres of the inscribed circle DEF and the escribed circle $D_1E_1F_1$ which touches BC and the other two sides produced. Join IE, I_1E_1 .

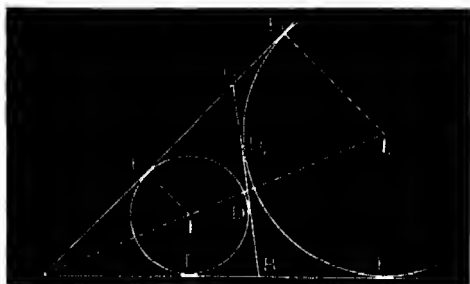


Fig. 56.

Then

$$\Delta^2 = r \cdot r_1 \cdot s \cdot (s-a) \dots\dots\dots \text{Expts. 71, 72,}$$

$$= s(s-a)(s-b)(s-c),$$

$$\begin{aligned} \sin^2 \frac{A}{2} &= \frac{r}{AI} \cdot \frac{r_1}{AI_1} \\ &= \frac{(s-b)(s-c)}{bc} \dots\dots\dots \text{Expts. 73, 74,} \end{aligned}$$

$$\begin{aligned} \cos^2 \frac{A}{2} &= \frac{AE}{AI} \cdot \frac{AE_1}{AI_1} \\ &= \frac{s(s-a)}{bc} \dots\dots\dots \text{Expt. 74,} \end{aligned}$$

$$\begin{aligned} \tan^2 \frac{A}{2} &= \frac{r}{AE} \cdot \frac{r_1}{AE_1} \\ &= \frac{(s-b)(s-c)}{s(s-a)} \dots\dots\dots \text{Expt. 73.} \end{aligned}$$

The "Sine Rule" can be used with logarithms to solve a triangle in which (i) *one side and two angles*, and (ii) *two sides and an opposite angle* are given: the following experiment will give us a formula for the remaining case in which *two sides and the included angle* are given.

77. Let ABC be a triangle in which $\angle A > \angle B$: with centre C and radius CA , describe a circle cutting BC in D and BC produced in E ; join AD , AE and draw DF parallel to EA to cut BA in F .

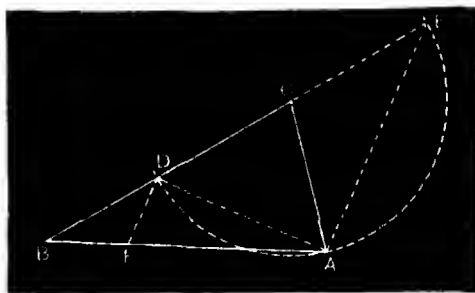


FIG. 57.

Then $\angle FDA = \angle DAE = \text{a rt. } \angle$.

Also $\angle EDA = \frac{1}{2} \angle ECA = \frac{1}{2} (A+B)$,

$\therefore \angle DAF = \angle EDA - \angle DBA = \frac{1}{2} (A-B)$;

and $BD = a-b$, $BE = a+b$.

Hence

$$\frac{\tan \frac{1}{2} (A-B)}{\tan \frac{1}{2} (A+B)} = \frac{DF}{DA} \cdot \frac{AE}{DA} = \frac{DF}{AE}.$$

But, since BDF , BEA are equiangular Δ s,

$$\therefore \frac{DF}{AE} = \frac{BD}{BE} = \frac{a-b}{a+b}.$$

$$\therefore \frac{\tan \frac{1}{2} (A-B)}{\tan \frac{1}{2} (A+B)} = \frac{a-b}{a+b},$$

$$\text{i.e., } \tan \frac{1}{2} (A-B) = \frac{a-b}{a+b} \cot \frac{1}{2} C.$$

EXERCISES.

1. Obtain the following expression for R , the radius of the circle circumscribed to the triangle ABC :—

$$(i) \quad R = \frac{a}{2 \sin A}, \quad (ii) \quad R = \frac{abc}{4\Delta}.$$

2. Verify the formulae

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},$$

$$r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

3. Shew that

$$r_1 + r_2 + r_3 - r = 4R,$$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

4. The sides of a right-angled triangle are 12, 16, 20: find r , r_1 , r_2 , r_3 , R .

5. The sides of a triangle are

$$(i) \quad 17, 25, 26;$$

$$(ii) \quad 11, 13, 20;$$

$$(iii) \quad 35, 44, 75;$$

$$(iv) \quad 13, 20, 21.$$

Find, in each case, the angles of the triangle and its area.

6. Two angles of a triangle are 59° , 61° ; the side adjacent to both is 10 inches long; find the difference in area between this triangle and an equilateral triangle on the same base.

7. The difference between two angles of a triangle is a right angle: the two sides opposite these angles are in the ratio of 2 : 1: find the angles of the triangle.

8. By means of the formulae given in Expt. 76 verify that

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 2 \cos^2 \frac{A}{2} - 1$$

$$= 1 - 2 \sin^2 \frac{A}{2}.$$

§ 18. Solution of Triangles and Problems.

By the use of seven-figure logarithms it can be shewn that the following are the six "parts" of a certain triangle ABC:—

$$a = 19828 \quad A = 28^\circ 33' 56.8'',$$

$$b = 37624 \quad B = 65^\circ 8' 20.8'',$$

$$c = 41380 \quad C = 86^\circ 17' 42.4''.$$

This triangle is solved from different data, by the three standard formulae, by the five-figure tables given on pp. 88—95, in Ex. 1—5. These worked-out exercises will serve, not only to shew the degree of accuracy of five-figure logarithms, but also as model solutions.

When calculating the "possible error," it should be remembered that the last figure in five-figure tables is thus determined from seven-figure tables. If the sixth and seventh figures taken together are less than 50, they are omitted; whilst if they are over 50, the fifth figure is increased by 1.

For example, consider the following:—"Find $\log 329.34$."

From seven-figure tables

$$\log 3.2900 = 0.5171959, \quad \log 3.3000 = 0.5185139,$$

\therefore with five-figure tables

$$\log 3.2900 = 0.51720, \quad \log 3.3000 = 0.51851,$$

and by interpolation

$$\log 3.2934 = 0.51765,$$

$$\therefore \log 329.34 = 2.51765.$$

Now with seven-figure tables it is found that

$$\log 329.34 = 2.5176445$$

$$= 2.51764 \text{ to five figures.}$$

Hence there is an error = 1 in the fifth figure.

Again, for example :—"Find $\log 1.1405$."

From seven-figure tables,

$$\log 1.1400 = 0.0569049, \quad \log 1.1500 = 0.0606978,$$

\therefore with five-figure tables,

$$\log 1.1400 = 0.05690, \quad \log 1.1500 = 0.06070,$$

and by interpolation

$$\log 1.1405 = 0.05709.$$

Now with seven-figure tables it is found that

$$\begin{aligned} \log 1.1405 &= 0.0570953 \\ &= 0.05710 \text{ to five figures,} \end{aligned}$$

and there is again an error of 1 in the fifth figure.

* Hence in the following exercises it may be taken that there is a *possible* error $= \pm 1$ in each logarithm taken, and supposing that the worst happens, i.e. that the error is of one sign for those logarithms which have to be added, and of the other sign for those which have to be subtracted in any particular problem, the greatest possible error is equal to the number of separate logarithms used in the calculation. Thus in using the formula

$$\log \tan^2 \frac{A}{2} = \log (s - b) + \log (s - c) - \log s - \log (s - a)$$

we *may* have an error $= \pm 4$ in the fifth figure; this would of course give a possible error $= \pm 2$ in the value of $L \tan \frac{A}{2}$.

NOTE. It should be noticed that the difference between the *five-figure* logarithms obtained (i) by interpolation and (ii) by direct approximation from seven-figure tables has been taken as the *greatest possible error*: these values are however generally one on each side of the true value, differing only by .5 in the fifth place from their true value, as in the examples given above.

1. Three sides.

Given $a=19828$, $b=37624$, $c=41380$; find by separate calculations the angles A, B, C.

[Model Solution.]

Formulae used.

$$(1) \tan^2 \frac{1}{2} A = \frac{(s-b)(s-c)}{s(s-a)},$$

$$(2) \tan^2 \frac{1}{2} B = \frac{(s-c)(s-a)}{s(s-b)},$$

$$(3) \tan^2 \frac{1}{2} C = \frac{(s-a)(s-b)}{s(s-c)}.$$

| | | |
|--------------|------------|----------------------|
| <u>Data.</u> | $a=19828$ | $\therefore s=49416$ |
| | $b=37624$ | $s-a=29588$ |
| | $c=41380$ | $s-b=11792$ |
| | $2s=98832$ | $s-c=8036$ |

Hence (i) $\log \tan^2 \frac{1}{2} A = \log 11792 + \log 8036 - \log 49416 - \log 29588$

$$\begin{array}{r}
 = 4.06819 \quad -4.69373 \\
 \quad \quad \quad 340 \quad \quad \quad 14 \\
 \quad \quad \quad 3.90472 \quad \quad \quad 4.46982 \\
 \quad \quad \quad 32 \quad \quad \quad 129 \\
 = 7.97663 \\
 -9.16498 \quad \leftarrow \\
 = 2.81165
 \end{array}$$

$$\therefore L \tan \frac{1}{2} A = 1.40583 + 10$$

$$\begin{array}{r}
 = 9.40583 \left\{ \begin{array}{l} \text{diff.} \\ 212 \end{array} \right. = 371 \\
 L \tan 14^\circ 10' \quad \quad \quad 212 \\
 \quad \quad \quad 20' \quad \quad \quad 742 \left\{ \begin{array}{l} \text{diff. for } 10' = 530, \\ \end{array} \right.
 \end{array}$$

$$\begin{aligned}
 \therefore \frac{1}{2} A &= 14^\circ 10' + \frac{371}{742} \times 10' \\
 &= 14^\circ 17', \\
 \therefore A &= 28^\circ 34'.
 \end{aligned}$$

NOTE. Here the possible error in $\log \tan^2 \frac{1}{2} A$ is .00004, leading to possible errors of .00032 in $L \tan \frac{1}{2} A$, of $2\frac{1}{2}''$ in $\frac{A}{2}$ (since diff. for $10' = 530$) and $5''$ in A. As the answer is only $3.2''$ wrong, some of the errors above have either not existed or cancelled one another. (Cf. Note, p. 110.)

Again (ii) $\log \tan^2 \frac{B}{2} = \log 8036 + \log 29588 - \log 49416 - \log 11792$

$$\begin{array}{r}
 = 3.90472 \qquad - 4.69373 \\
 32 \qquad 14 \\
 4.46982 \qquad 4.06819 \\
 129 \qquad 340 \\
 \hline
 = 8.37615 \\
 - 8.76546 \quad \leftarrow \\
 \hline
 = 1.61069
 \end{array}$$

$$\begin{array}{lcl}
 \therefore L \tan \frac{B}{2} = 9.80535 & \left. \begin{array}{l} \text{diff.} \\ 419 \end{array} \right\} & = 116. \\
 L \tan 32^\circ 30' & & \\
 40' & \left. \begin{array}{l} \text{diff. for } 10' = 278. \\ 697 \end{array} \right\} &
 \end{array}$$

$$\therefore \frac{B}{2} = 32^\circ 34' 10'',$$

$$\therefore B = \underline{65^\circ 8' 20''}.$$

Similarly (iii) $\log \tan^2 \frac{C}{2} = 1.94379;$

$$\therefore L \tan \frac{C}{2} = 9.97190,$$

$$\therefore C = 86^\circ 17' 42''.$$

Hence we have

$$A = 28^\circ 34'$$

$$B = 65^\circ 8' 20''$$

$$C = 86^\circ 17' 42''$$

$$\underline{180^\circ 0' 2''}.$$

2. Two sides and the included angle.

Given $b = 37624$, $c = 41380$, $A = 28^\circ 33' 56.8''$; find B , C and a .

[*Model Solution.*]

Formulae used

$$(1) \tan \frac{1}{2}(C - B) = \frac{c - b}{c + b} \tan \frac{1}{2}(C + B).$$

$$(2) \quad a = \frac{b \sin A}{\sin B}.$$

CASE II. TWO SIDES, INCLUDED ANGLE 113

Data.

$$\begin{aligned} c &= 41380 & A &= 28^\circ 33' 56.8'' \\ b &= 37624 & \therefore C+B &= 151^\circ 26' 3.2'' \\ \therefore c-b &= 3756 & \therefore \frac{1}{2}(C+B) &= 75^\circ 43' 1.6''. \\ c+b &= 79004. \end{aligned}$$

Hence (i) $L \tan \frac{1}{2}(C-B) = \log 3756 + L \tan 75^\circ 43' 1.6'' - \log 79004$

$$\begin{aligned} &= \begin{array}{r} 3.57403 \\ 70 \\ 10.59258 \\ 160 \\ \hline 14.16891 \end{array} - \begin{array}{r} 4.89763 \\ 2 \\ \hline 4.89765 \end{array} \\ &= \begin{array}{r} 9.27126 \\ 6797 \\ 40' \quad 7496 \end{array} \left. \begin{array}{l} \text{diff.} \\ \text{diff. for } 10' \end{array} \right\} \begin{array}{l} = 329. \\ = 699. \end{array} \\ \therefore \frac{1}{2}(C-B) &= 10^\circ 30' + \frac{329}{699} \times 10' \\ &= 10^\circ 34' 42.3''. \end{aligned}$$

But $\frac{1}{2}(C+B) = 75^\circ 43' 1.6''$,
 $\therefore C = 86^\circ 17' 43.9''$
 $B = 65^\circ 8' 19.3''$.

Again (ii) $\log a = \log 37624 + L \sin 28^\circ 33' 56.8'' - L \sin 65^\circ 8' 19.3''$.

$$\begin{aligned} &= \begin{array}{r} 4.57519 \\ 28 \\ 9.67866 \\ 91 \\ \hline 14.25504 \end{array} - \begin{array}{r} 9.95728 \\ 48 \\ \hline 9.95776 \end{array} \\ &= \begin{array}{r} 4.29728 \\ 667 \\ 9 \quad 885 \end{array} \left. \begin{array}{l} \text{diff.} \\ \text{diff. for } 1 \end{array} \right\} \begin{array}{l} = 61. \\ = 218. \end{array} \\ \log 1.98 & \\ \therefore a &= 1.98 \frac{61}{218} \times 10^4 \\ &= \underline{19828}. \end{aligned}$$

NOTE. In the last example the possible error in $L \tan \frac{1}{2}(C-B)$ is .00008; this, in that part of the table corresponding to the approximate value of $\frac{1}{2}(C-B)$, where the difference for $10'$ is 699, corresponds to a possible error of less than $3''$ in $\frac{1}{2}(C-B)$, and hence in C and in B . This error might affect the finding of a , but the difference for $10'$ in L sines between 65° and 66° is less than 60, and hence an error of $3''$ will not affect the figure in the fifth decimal place.

3. Two sides and the angle opposite the smaller of these. (*The ambiguous case.*)

Given $a=19828$, $c=41380$, $A=29^\circ 33' 56.8''$; solve the triangle.

[*Model Solution.*]

$$\text{Formula used.} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\begin{array}{ll} \text{Data.} & a=19828 \quad A=28^\circ 33' 56.8'' \\ & c=41380. \end{array}$$

$$\text{Hence} \quad L \sin C = \log 41380 - \log 19828 + L \sin 28^\circ 33' 56.8''$$

$$= 4.61595 \quad - 4.29667$$

$$\quad \quad \quad 84 \quad \quad \quad 61$$

$$\quad \quad \quad 9.67866$$

$$\quad \quad \quad 91$$

$$= 14.29636$$

$$- 4.29728$$

$$\quad \quad \quad 9.99908 \left\{ \begin{array}{l} \text{diff.} = 5 \\ \text{diff. for } 10' = 8. \end{array} \right.$$

$$L \sin 86^\circ 10' = 3$$

$$20' = 11$$

$$\therefore C \text{ or } 180^\circ - C = 86^\circ 10' + \frac{5}{8} \times 10'$$

$$= 86^\circ 16' 15''$$

$$\therefore C = \underline{86^\circ 16' 15''} \text{ or } \underline{93^\circ 43' 45''}.$$

Case I. If

$$C = 86^\circ 16' 15'',$$

Then,

$$\therefore A = 28^\circ 33' 57'' \text{ (approx.)}$$

$$\therefore B = 65^\circ 9' 48''$$

$$\underline{180^\circ 0' 0''}$$

CASE III. TWO SIDES, ONE OPPOSITE ANGLE 115

Hence $\log b = \log 19828 + L \sin 65^\circ 9' 48'' - L \sin 28^\circ 33' 57''$.

$$\begin{array}{rcl}
 & = & 4.29667 \quad -9.67866 \\
 & & \begin{array}{r} 61 \\ 9.95728 \\ 55 \end{array} \quad \begin{array}{r} 91 \\ \hline \end{array} \\
 & = & 14.25511 \\
 & - & 9.67957 \quad \leftarrow \\
 & = & 4.57554 \left. \begin{array}{l} \text{diff.} \\ 519 \\ 634 \end{array} \right\} \begin{array}{l} = 35, \\ \text{diff. for } 1 = 115, \end{array} \\
 \log 3.76 & & \\
 7 & & \\
 \therefore b & = & 3.76_{115} \times 10^4 \\
 & = & \underline{37630}.
 \end{array}$$

Case II.

If $C = 93^\circ 43' 45''$

Then,

$\therefore A = 28^\circ 33' 57''$ (approx.)

$\therefore B = 57^\circ 42' 18''$
 $180^\circ 0' 0''$

$\therefore \log c = \log 19828 + L \sin 57^\circ 42' 45'' - L \sin 28^\circ 33' 57''$.

$$\begin{array}{rcl}
 & = & 4.29667 \quad -9.67866 \\
 & & \begin{array}{r} 61 \\ 9.92683 \\ 22 \end{array} \quad \begin{array}{r} 91 \\ \hline \end{array} \\
 & = & 14.22433 \\
 & - & 9.67957 \quad \leftarrow \\
 & = & 4.54476 \left. \begin{array}{l} \text{diff.} \\ 407 \\ 531 \end{array} \right\} \begin{array}{l} = 69, \\ \text{diff. for } 1 = 124, \end{array} \\
 \log 3.50 & & \\
 1 & & \\
 \therefore c & = & 3.50_{124} \times 10^4 \\
 & = & \underline{35056}.
 \end{array}$$

NOTE. The possible error of .00003 in $L \sin C$ corresponds to $\frac{1}{8}$ of $10'$, i.e. about $4'$ in C . In general, the results obtained from L sines when the angles are near 90° are very untrustworthy. This affects the values of b , and in such cases the more laborious method, of using the formula $a^2 = b^2 + c^2 - 2bc \cos A$ to find b first, should be adopted. C can then be found from

$$\tan^2 \frac{1}{2} C = \frac{(s-a)(s-b)}{s(s-a)}.$$

4. Two sides and the angle opposite the greater of these.

Given $a=19828$, $b=37624$, $B=65^\circ 8' 20.8''$; find A .

[Model Solution.]

Formula used. $\sin A = \frac{a}{b} \sin B.$

Data. $a=19828$ $B=65^\circ 8' 20.8''.$
 $b=37624.$

Hence $L \sin A = \log 19828 - \log 37624 + L \sin 65^\circ 8' 20.8''$

$$\begin{array}{rcl}
 & = & 4.29667 \quad -4.57519 \\
 & & \quad \quad \quad 61 \quad \quad \quad 28 \\
 & & 9.95728 \\
 & & \quad \quad \quad 48 \\
 & = & 14.25504 \\
 & - & 4.57547 \quad \leftarrow \\
 & = & 9.67957 \\
 L \sin 28^\circ 30' & = & 7866 \quad \left. \begin{array}{l} \text{diff} \\ \text{diff. for } 10' = 232, \end{array} \right\} \text{diff} = 91. \\
 40' & & 8098 \\
 \therefore A & = & 28^\circ 30' + \frac{91}{232} \times 10' \\
 & = & \underline{28^\circ 33' 55'' *}.
 \end{array}$$

NOTE. Here the possible error in $L \sin A$ may be .00003 which corresponds to a possible error of over 6'' in A .

* The supplementary value $151^\circ 26' 1''$ is inadmissible for $A+B$ would then be equal to $216^\circ 84' 21.8''$, which is impossible.

5. One side and two angles.

Given $a=19828$, $B=65^\circ 8' 20.8''$, $C=86^\circ 17' 42.4''$; solve the triangle.

[Model Solution.]

Formula used. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$

Data. $a=19828.$ $B=65^\circ 8' 20.8''$
 $C=86^\circ 17' 42.4''$
 $\therefore A=28^\circ 33' 56.8''.$

CASE IV. ONE SIDE, TWO ANGLES **117**

Hence (i) $\log b = \log 19828 + L \sin 65^\circ 8' 20.8'' - L \sin 28^\circ 33' 56.8''$.

$$\begin{array}{rcl}
 & = & 4.29667 \qquad - 9.67866 \\
 & & \begin{array}{r} 61 \\ 9.95728 \\ 48 \\ \hline 14.25504 \\ - 9.67957 \\ \hline 4.57547 \end{array} \qquad \begin{array}{r} 91 \\ \hline \end{array} \\
 & = & \begin{array}{r} 14.25504 \\ - 9.67957 \\ \hline 4.57547 \end{array} \left\{ \begin{array}{l} \text{diff.} \\ \text{diff. for } 1 = 115, \end{array} \right. \begin{array}{r} = 28, \\ \\ \end{array} \\
 \log 3.76 & & \begin{array}{r} 519 \\ 7 \quad 634 \end{array} \\
 \therefore b & = & 3.76_{115} \times 10^4 \\
 & = & \underline{\underline{37624.}}
 \end{array}$$

Again (ii) $\log C = \log 19828 + L \sin 86^\circ 17' 42.4'' - L \sin 28^\circ 33' 56.8''$

$$\begin{array}{rcl}
 & = & 4.29667 \qquad - 9.67866 \\
 & & \begin{array}{r} 61 \\ 9.99903 \\ 6 \\ \hline 14.29637 \\ - 9.67957 \\ \hline 4.61680 \end{array} \qquad \begin{array}{r} 91 \\ \hline \end{array} \\
 & = & \begin{array}{r} 14.29637 \\ - 9.67957 \\ \hline 4.61680 \end{array} \left\{ \begin{array}{l} \text{diff.} \\ \text{diff. for } 1 = 105, \end{array} \right. \begin{array}{r} = 85. \\ \\ \end{array} \\
 \log 4.18 & & \begin{array}{r} 595 \\ 4 \quad 700 \end{array} \\
 \therefore c & = & 4.13_{105} \times 10^4 \\
 & = & \underline{\underline{41381.}}
 \end{array}$$

6. Find the area of the triangle whose sides are 384.11 ft., 564.67 ft., 663.44 ft., in acres, roods and poles to the nearest pole.

7. The sides of a triangle are 13, 9, 12; find the greatest angle of the triangle.

8. Given $b=105$, $c=55$, $A=51^\circ$, find B and C.

9. Given $a=1770.1$, $b=2164.5$, $A=35^\circ 36' 20''$; find B, C and c.

10. Find all the parts of the triangles which have one side 90 ft. long, another side 60 ft. long, and the angle opposite to the shorter side equal to $18^\circ 37'$.

11. Two sides of a triangle are 200 ft. and 300 ft. respectively, the area is 20,000 square feet, calculate the angle between the two sides. Explain the two results that are obtained. Calculate the remaining side in each case.

12. From two stations A and B on shore, 3,742 yds. apart, a ship C is observed out at sea. The angles BAC, ABC are simultaneously observed to be $72^{\circ} 34'$ and $81^{\circ} 41'$ respectively. Find the distance of the ship from A.

13. From the top of a vertical cliff 100 ft. high, forming one bank of a river, the angles of depression of the top and bottom of a vertical cliff forming the opposite bank are $28^{\circ} 40'$ and $64^{\circ} 30'$ respectively. Find the height of the cliff on the opposite bank, and the breadth of the river.

14. P is a point vertically over N, a point in a horizontal plane, A and B are two points in the plane; AN is 100 ft., and the angles of elevation of P at A and B are $24^{\circ} 30'$ and $8^{\circ} 10'$ respectively: find the distance BN. [First find PN.]

15. AB is a line 250 ft. long, in the same horizontal plane as the foot, D, of a tower CD; the angles DAB and DBA are respectively $61^{\circ} 23'$ and $47^{\circ} 14'$; the angle of elevation of C from A is $34^{\circ} 50'$; find the height of the tower.

16. A and B are two places on opposite sides of a mountain; C is a distant landmark visible from A and B, AC=10 miles, BC=8 miles, the elevations of A and B at C are 8° and $2^{\circ} 48'$ respectively, and the angle BCA is $63^{\circ} 28'$. A tunnel has to be bored from A to B. Calculate

- (1) the difference in height between A and B;
- (2) the angle it makes with the horizon;
- (3) the angle it makes with the direction AC;
- (4) its length to nearest yard.

§ 19. Surveying Instruments.

As has been pointed out in § 2, a protractor whose radius is four inches, graduated in degrees, has the divisions one-fifteenth of an inch long: these could easily be divided into fifths, or odd fifths might be estimated with fair accuracy by eye. Errors of drawing, and even the breadth of the pen or pencil lines used, make it useless to strive for any greater degree of accuracy when solving problems graphically.

Now since in surveying and astronomical work generally, problems are solved by calculations with tables—which can be calculated to any degree of accuracy—accuracy of solution is only limited by the degree of accuracy with which measurements can be made.

By means of machinery, the protractor referred to above could be graduated into minutes or even seconds of arc; but the lines would have to be so excessively fine and close together, that the scale would be practically useless, even with a magnifying glass or low power “reading microscope.” This excessive subdivision of a straight or circular scale is obviated by an auxiliary scale, sliding in contact with the main scale; this auxiliary scale is called a **vernier** after its inventor.

Verniers are of two kinds, “forward-reading” and “backward-reading.” The principle is the same for both.

The value of the vernier in improving accuracy of measurements depends on the fact that the eye detects with considerable accuracy whether two straight lines are
* in exact alignment or not,

The "Forward-reading Vernier."

The simplest form of vernier, and the usual one for straight scales, is the *decimal* vernier.

Let **AB** be a scale of inches and tenths, let **V** be an auxiliary scale, $\frac{9}{10}$ in. long, divided into ten equal parts: then the difference between a scale division and a vernier division is

$$\frac{1}{10} - \frac{1}{10} \cdot \frac{9}{10} = \frac{1}{100} \text{ in.}$$

Hence, if any vernier division is opposite to a scale division, moving the vernier $\frac{1}{100}$ in. to the right will bring the next vernier division opposite a scale division. Hence, if the vernier divisions are numbered consecutively from left to right, the number attached to a division on the vernier, which comes opposite to (or most nearly so) a scale division, gives the number of hundredths of an inch which the vernier has been moved from the position in which its left-hand edge came opposite a scale division. Thus, in Fig. 58, the length of the pencil is equal to 3.6 inches + the amount the vernier has been moved to the right from the position in which the left-hand edge was opposite "3.6" on the scale. Since the division marked 4 on the vernier is opposite a scale division, it is seen that the vernier has been moved to the right $4 \times .01$ in. Hence the length of the piece of pencil is

$$3.64 \text{ inches.}$$

It will readily be seen that the vernier zero need not be at the edge of the auxiliary scale, so long as it is distant from it an exact number of tenths of an inch; and similarly for the other end of the vernier.

1. Read the height of the barometer, in Fig. 59, (i) to .002 in. (right-hand side), (ii) to .1 mm. (left-hand side).

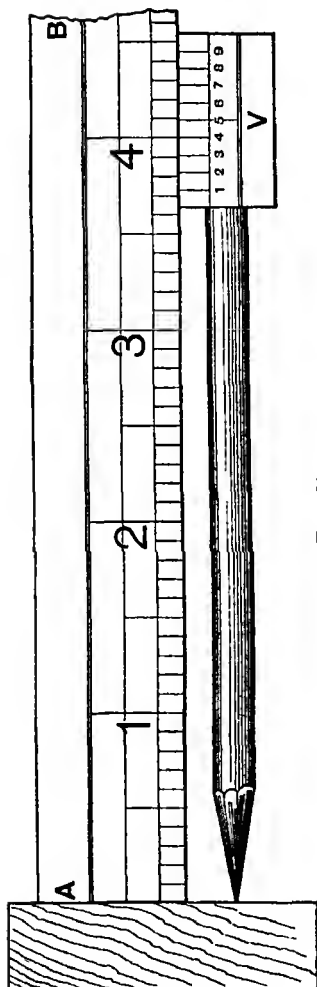


FIG. 58.

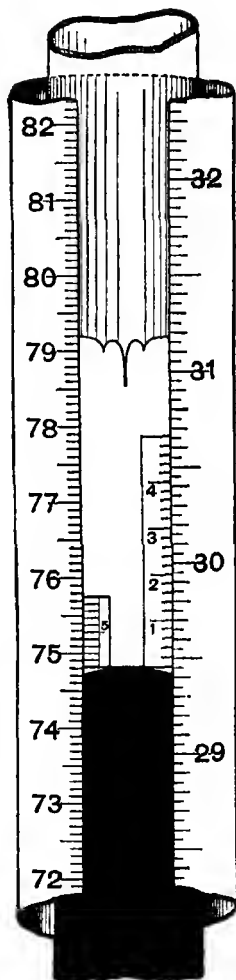


FIG. 59.

Verniers, on the same principle as the decimal vernier, may be constructed, in which any arbitrary number (n) of divisions may be taken from the scale and the total length of these n divisions subdivided on the vernier into $(n+1)$ equal parts, the vernier so constructed reading to $\frac{1}{n+1}$ th of a scale subdivision.

For instance, with a scale in inches and tenths, take 19 tenths on a vernier divided into 20 equal parts: then difference between a scale division and a vernier division is

$$\frac{1}{10} - \frac{1}{20} \cdot \frac{19}{10} = \frac{1}{200},$$

and the vernier reads to .005 inch, *although the distance between the graduations on the vernier is very little less than one-tenth of an inch.*

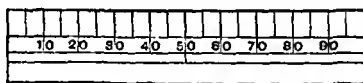


FIG. 60.

Fig. 60 shews how this vernier should be numbered; the division 85 on the vernier, for instance, if coincident with a subdivision on the scale, standing for an addition of .085 in., to the inches and tenths, observed directly from the scale, in the length to be measured.

With such a vernier a reading microscope would be necessary; for with the naked eye, there would be great difficulty in detecting which, out of a group of several consecutive lines on the vernier, was the one in most accurate alignment with a division on the scale.

2. Construct a vernier for use with a scale of inches and eighths to read to a thirty-second of an inch [here $n=3$].

Although *theoretically* a scale with a sufficiently long vernier can thus be constructed to read to any degree of accuracy, *practically* there is a limit imposed owing to the fact that the number of vernier lines which are in approximate alignment with corresponding scale divisions becomes greater as the difference between the subdivisions on the vernier and scale gets less.

The "Backward-reading Vernier."

If n scale divisions are taken and divided into $n-1$ equal parts; then each division on the vernier is to the right of the corresponding scale division by $\frac{1}{n-1}$ of a scale division, if we count as before from the left-hand end of the vernier scale; if however we count from the right-hand end, the vernier divisions are to the left of the corresponding scale divisions as before.

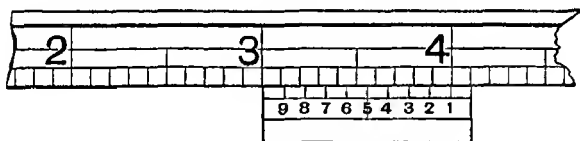


FIG. 61.

Thus, Fig. 61 shows a scale in inches and tenths with a vernier, on which eleven-tenths of an inch have been taken and divided into 10 equal parts, and numbered as shown. This scale and vernier read to '01 in.

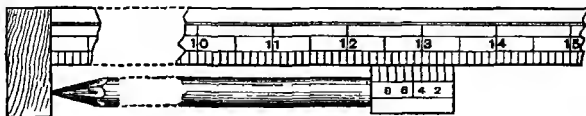


FIG. 62.

- In Fig. 62 the length of the pencil is 12.32 cm.

***General theory of the vernier.**

Let AB be part of a scale graduated in units and sub-units; let there be m sub-units in a whole unit.

Let CD, the vernier, be another scale capable of sliding with its edge always in contact with the edge of the scale AB.

Mark off on the vernier a distance equal to n sub-units of the scale, and subdivide this distance into $p(>n)$ equal parts.

Then the distance between any two marks on the vernier is equal to $\frac{1}{p} \cdot \frac{n}{m}$ units of scale.

\therefore difference between the subdivisions on the scale and the vernier is equal to

$$\frac{1}{m} - \frac{1}{p} \cdot \frac{n}{m} = \frac{p-n}{p \cdot m} \text{ units.}$$

Although p can be any number greater than n , for convenience it is almost always taken equal to $n+1$, thus making the difference between the subdivisions equal to $\frac{1}{pm}$ units.

In consequence, the r th division of the vernier is $\frac{r}{pm}$ units to the left of the corresponding scale division; and, since $\frac{r}{pm}$ is never greater than $\frac{1}{m}$, there is no ambiguity caused by the r th vernier division falling past the $(r-1)$ th division of the scale.

The importance of this may be more readily seen by considering the following example.

A scale is divided into units and fifths: on the vernier a length equal to three-fifths of a unit is divided into eight parts.

Each division on the vernier is therefore equal to $\frac{3}{40}$ ths of a unit. Hence the first vernier division falls to the left of the nearest scale division by $\frac{1}{5} - \frac{3}{40} = \frac{5}{40}$ unit;

| | |
|------------------------------|---|
| the second vernier division, | by $\frac{2}{5} - \frac{6}{40} = \frac{4}{40}$ unit; |
| the third | by $\frac{3}{5} - \frac{9}{40} = \frac{3}{40}$ unit; |
| the fourth | by $\frac{4}{5} - \frac{12}{40} = \frac{2}{40}$ unit; |
| the fifth | by $\frac{5}{5} - \frac{15}{40} = \frac{1}{40}$ unit; |
| the sixth | by $\frac{6}{5} - \frac{18}{40} = \frac{2}{40}$ unit; |
| the seventh | by $\frac{7}{5} - \frac{21}{40} = \frac{3}{40}$ unit. |

* These two pages may be omitted on first reading.

Hence the vernier must be numbered as in Fig. 63: and, if any division of the vernier coincides with a division of the scale, it shews that the vernier has been moved a distance

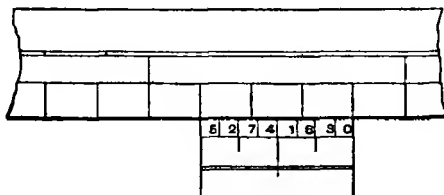


FIG. 63.

equal to r -fortieths of a unit (where r is the number attached to the vernier division in coincidence) to the right from the position in which the vernier zero coincided with a scale division.

If however $p=n+1$, the numbering is consecutive from the zero from left to right, as in the decimal vernier.

Suppose we have to measure a rod between 4 and 5 units long, and that when one end is placed exactly opposite the zero of the scale, the other end comes between the sixth and seventh subdivisions on the scale; also that, when the vernier (having previously been pushed along out of the way) is brought back with its zero end into contact with the rod, it is found that the r th division on the vernier coincides with a scale division. Then the length of the rod is 4 units + 6 subdivisions + the distance the vernier zero has been moved from a position of coincidence with the sixth scale subdivision. Now in this latter position the distance of the r th vernier division from the nearest scale division to the right of it is $\frac{r}{pm}$ units; hence if these division marks now coincide the vernier must have been moved a distance to the right equal to $\frac{r}{pm}$ units.

Hence the length of the rod is

$$4 + \frac{6}{m} + \frac{r}{pm} \text{ units.}$$

The principle is the same for the measurement of angles, the scale and vernier being annular and concentric: if a degree is divided into m equal parts on the scale, and if a length equal to n of these on the vernier are divided into p ($=n+1$) equal parts, the scale and vernier will read to $\frac{1}{pm}$ of a degree.

Thus a 5-inch circle might be graduated at intervals of $5'$, the intervals being about $\frac{1}{100}$ inch long: on the vernier, about $1\frac{1}{2}$ inch long, a length equal to 149 of these intervals might be taken and divided into 150 equal parts: the scale and vernier would read to

$$\frac{1}{100} \cdot \frac{5}{150} \text{ of a degree} = 2'' \text{ of angle.}$$

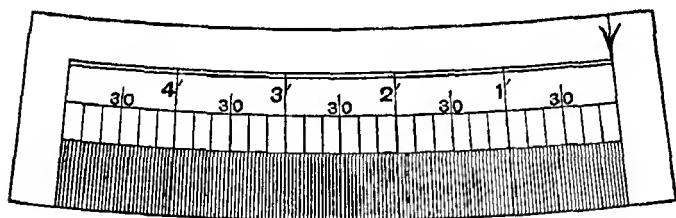


FIG. 64.

A reading microscope would of course be necessary: the vernier should be figured as in Fig. 64, which is enlarged to about twice natural size.

3. How would you divide a scale and vernier on a 4-inch circle to read to half-minutes, so that the vernier is not more than an inch long?

4. What is the radius of the smallest circle, which, with a vernier not more than 2 inches long, and the divisions on both scale and vernier not less than .01 inch apart, will read to seconds?

The Surveyor's Level.

This instrument consists of a telescope mounted on a stand; it is fitted with levelling screws and two ordinary pattern "levels," one of which is, when accurately adjusted, parallel to the axis of the telescope and the other at right angles to it.

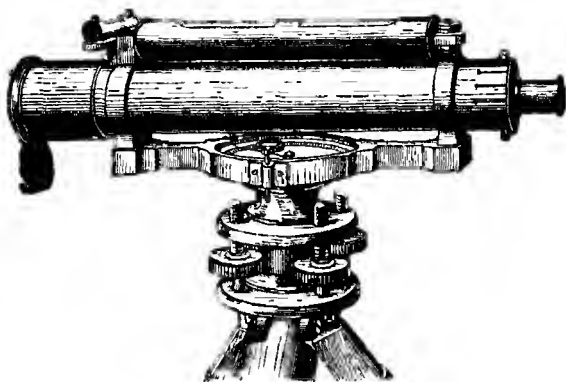


FIG. 65.*

A "compass" is fixed centrally under the telescope.

The Theodolite.

The simple theory of this instrument has been given in the introductory section. The explanation there given applies to the apparently complicated instrument, the '*Transit*' Theodolite (Fig. 67), the only essential difference being that the graduated vertical circle is rigidly fixed to the telescope (which corresponds with the pointer of Fig. 15), and turns with it, the rotation being measured against a fixed mark attached to the bearings of the telescope.

* From an instrument supplied by Messrs Griffin, London.

No verbal description, however well illustrated, can in any way take the place of a practical examination of such instruments as a theodolite, but it may enable the student to know what to look for, and where to find it, when he has the instruments in his hands; even though it is different in some small detail, such as disposition of the compass, different styles of locking-plates, by which the instrument is affixed to the stand, or different kinds of levelling-screw arrangement. A careful comparison of the figures of the theodolites on pp. 19, 131, and the level on p. 127 will be of service. A particularly handy arrangement for levelling the parallel plates, the upper of which carries the mounting of the telescope, and for centering the instrument above a given peg in the ground is shewn on the instrument in Fig. 67. This is a variation of the "Hoffman Tripod Head;" of which Fig. 66 shews an elevation and a section through the axis, giving

all the essential features. It will readily be seen that the arrangement gives great freedom of adjustment. When the four milled levelling screws (of which two are shewn at AA) are all loosened, the upper part of the instrument can move round the centre of the lower ball X as a centre, in any direction, and also the centre of this

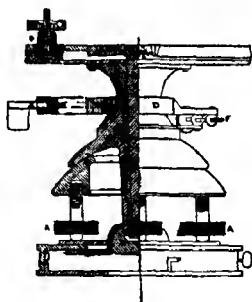


FIG. 66.

lower ball and its socket and plate attached can be moved laterally in any direction. Thus, the instrument can be rapidly centred over a given mark, and roughly levelled. The milled screws are then turned, forcing the upper ball into its socket and locking the whole combination by friction.

If now one pair of opposite screws be taken, releasing one of the pair and tightening the other, cant the whole of the top, until the axis of the ball and socket combination lies in the vertical plane containing the axes of the two screws used. Similar adjustment with the other pair of screws will then bring the axis exactly vertical. This is shewn by the pair of levels at right angles described below, one level being temporarily set parallel to each of the vertical planes containing the axes of a pair of levelling screws.

The axis of the ball and socket combination, now exactly vertical, is also the common axis of the two cones (B, C); to which the parallel plates are rigidly attached, at right angles to the common axis, the inner cone bearing the upper plate. Reference to Fig. 66 will shew that both plates can be freely rotated in a horizontal plane. The lower plate is rotated roughly into any required position, clamped to the ball and socket combination by means of the collar (DE), the tangent screw (F) providing for fine adjustment. The inner cone and upper parallel plate still have perfect freedom of rotation round the vertical axis; and after the upper plate has been placed roughly in any desired position, it is clamped firmly to the lower plate by means of the clutch (G), shewn in sectional elevation in Fig. 66. Fine adjustment is obtained by means of a tangent screw working against this clutch, the details being given (N) in Fig. 67 *a*. The lower plate is graduated, part of the scale being shewn at S in Fig. 66, and the upper plate bears a vernier V. Another vernier is generally added, diametrically opposite, to obviate errors of centring. Both are provided with reading microscopes.

The upper parallel plate (Fig. 67) carries a level (H), the fine adjustment for the horizontal circle, and the pillars

upon which the telescope is mounted. One of these pillars carries another level (K) accurately at right angles to the first, both levels being permanently adjusted parallel to the upper plate.

The telescope is mounted on a stout axle, at right angles to its "line of sight," this axle also carries the graduated vertical circle (LL) rigidly attached to it; the verniers (JJ) are attached to the arms of a T-piece, which can rotate freely round the axle of the telescope; its tail (W) being clamped to a cheek (X) on one of the pillars, and finely adjusted, by means of two screws (YY) working against one another; this arrangement is shewn in detail in Fig. 67 *b*. These screws are manipulated until the verniers both indicate zero*, when the telescope is pointing truly level, as indicated by the long delicate level (MM) attached to it parallel to its axis or "line of sight." When mounted the telescope can be roughly rotated by hand until the object to be observed is in the field of view. The clamp and tangent screw (N, O) combinations are then used to rotate the telescope horizontally and vertically, until the object comes accurately on the cross-wires, fixed in the telescope tube, which indicate the exact centre of the field of view.

Certain adjustments are called "permanent": these are generally manipulated by capstan screws as being less liable to be disarranged. The bearing at the top of one of the pillars is movable, allowing that end of the telescope axle to be raised or lowered, so as to render the axle perfectly* horizontal; frequently also there is a lateral motion at one end, bringing the axis of the telescope into a certain adjustment with the case of the compass. Other permanent adjustments are those of the three levels and the "line of sight or collimation" of the telescope. These should be frequently tested and set right if necessary.

* That is, if the circle and T-piece are exactly centred: if not the error is noted.

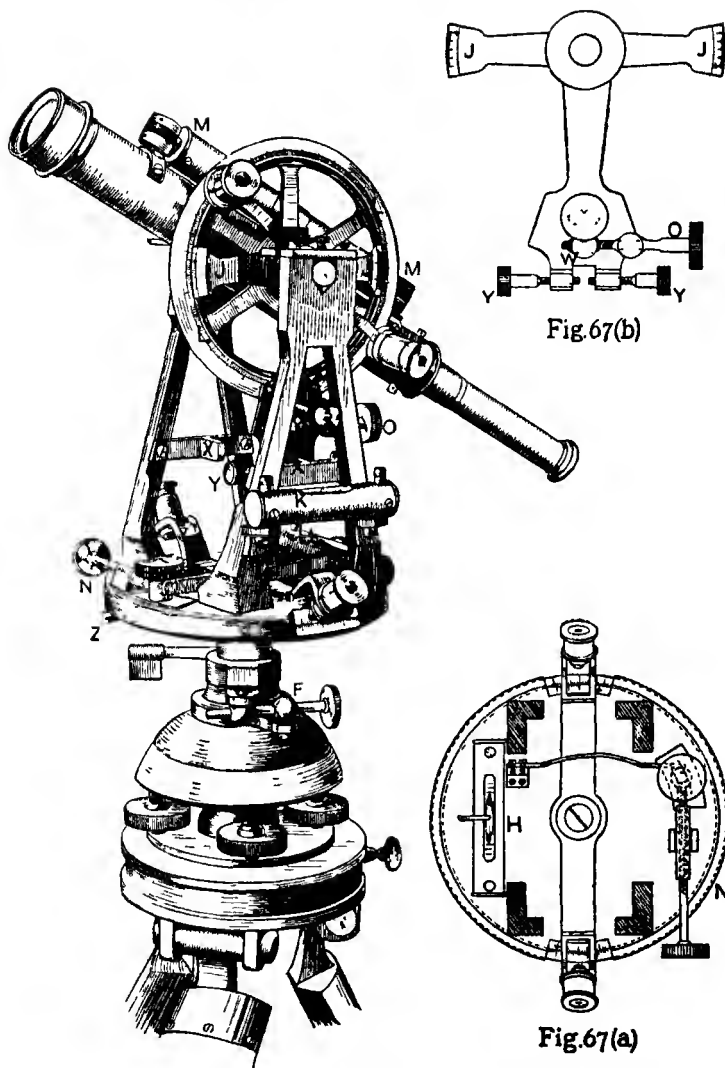


FIG. 87.

To prepare for an observation.

(i) Set up the tripod, getting the top fairly level and approximately over the ground-mark.

(ii) Fix on the tripod head, by means of the bayonet catch and thumbscrew, or the locking plates, or other device.

(iii) Manipulate the levelling screws until the parallel plates are perfectly horizontal, and the axis of the instrument vertically over the peg, as indicated by the plumb-line attached, and clamp and adjust the lower plate.

(iv) Test the permanent adjustments for level by swinging the upper plate round into various positions and observing the readings of the levels.

To take an observation of the difference in bearing of two objects P and Q, and of the elevations or depressions of each.

(i) Move the telescope and upper plate until P appears in the field of view.

(ii) Clamp the horizontal and vertical circles, and by means of the tangent screws bring P exactly on the cross-wires.

(iii) Read the verniers for both circles and loose the clamps.

(iv) Repeat the operations for Q.

(v) Read the verniers on vertical circle when the level attached to the telescope shews that it is horizontal.

The difference in the readings of the vertical circle

in (iii) and (v) give the elevation or depression of P; of those in (iv) and (v) give the elevation or depression in Q; whilst the difference in the readings of the horizontal circle in (iii) and (iv) give the difference in bearing of P and Q. The compass in the instrument illustrated in Fig. 67*, being fixed in slots (Z) beneath the lower plate, may be so adjusted that the horizontal readings give the true bearing of each object observed.

* This illustration is from a photograph of an instrument supplied by Messrs J. Davis, Derby and London.

The Sextant.

The sextant is an instrument for measuring the angular altitude or elevation of an object or, in general, the angle subtended at the observer's eye by any two distant objects. Its use is almost entirely confined to observations at sea, for which an instrument like a theodolite, depending on a fixed horizon, would be unsuitable; whilst the sextant (by reason of the principle of its construction, which brings two *very distant* objects *simultaneously* into the field of view of the telescope) is independent of the rolling and pitching of the ship, its progressive motion, or of any fixed base.

Before the student can grasp the principle of the sextant, there are four experimental facts connected with Optics which he must understand.

I. A convex lens can be used for two things according to its position with regard to an object:—

(a) As a magnifying glass.

(b) To form a picture, called the *image*, of an object, such as is seen on the ground-glass screen in a camera.

By a suitable adjustment of the distance between them, two convex lenses can be so arranged that one, called the object-glass (O),

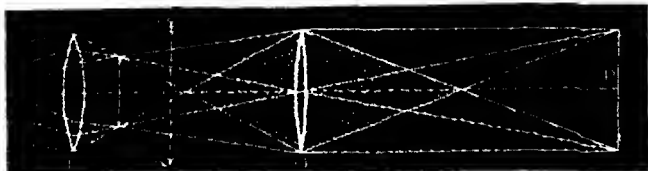


FIG. 68.

forms a reduced image *Q* of an object *P*, and the other, called the eye-piece (*E*), acts as a magnifying glass to observe this image. When the object is very distant, such as a star, the rays of light entering the object-glass are approximately parallel, and the image is formed at a point, at a fixed distance behind this lens, called the **focus**. It is at this point that the cross-wires of the telescope in a theodolite are fixed, so that they lie in the same plane as the image.

II. When half of the object-glass of a telescope is covered over, then an eye looking through the telescope will see exactly what it saw before except that the image is half as bright. This is readily seen from Fig. 69, where the rays proceeding from *P*, those entering the top half of the object-glass, are alone shown.

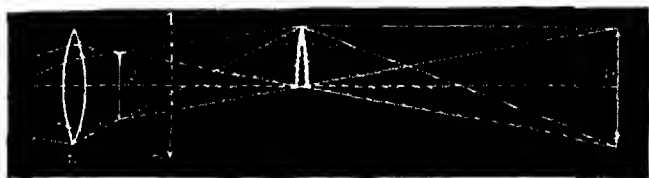


FIG. 69.

III. If a ray of light *ABA'*, travelling in a fixed direction in the plane of the paper, in Fig. 70, is incident on a plane mirror set up

perpendicular to the paper along PQ , and is reflected along the path BC , the angles ABP , CBQ are equal.

$$\therefore \angle A'BC = 2\angle A'BP.$$



FIG. 70.

Again, if the mirror is rotated into the position denoted by $P'Q'$, the reflected ray will travel along the path BC' , such that

$$\angle A'BC' = 2\angle A'BP'.$$

$$\therefore \angle CBC' = 2\angle PBP'.$$

Also, if a fixed mirror XY had been used to reflect a ray, originally travelling in the fixed direction OA , along the fixed direction AB , it is evident that when PQ is parallel to XY , BC is parallel to OA . Hence the angle PBP' is half the angle between BC' and OA .

IV. The path of a ray of light is reversible. Hence, if a telescope is placed with its axis pointing along OA , and the mirror is rotated until the ray, from a distant object, travelling in the direction $C'B$, after being twice reflected, enters the telescope, then the angle through which the mirror has been rotated, from the position in which it was parallel to XY , is half the angle between BC and OA .

Suppose now that the mirror XY is replaced by a glass plate, of which only the half AY is silvered. If OA points towards an object S' , an image of S' is formed in the telescope at i , whose brightness is half what it would be if the "mirror" XY was not there.

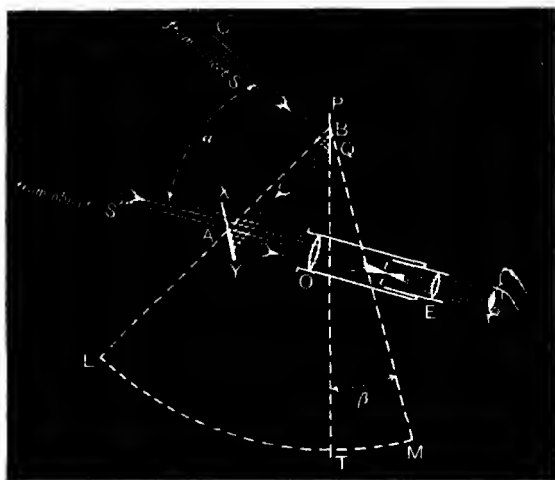


FIG. 71.

At the same time, an image of an object S , lying in the direction BC' , may be formed in the telescope at i , by reflection at the surfaces of PQ and the half mirror AY ; these two images may be brought into contact in the case of the sun or star and the horizon, or into superposition in the case of two stars, since both are formed at the principal focus of the object-glass. The angle TBM will then measure half the angle subtended by the two objects at the observer's eye. This is shewn in Fig. 71.

The following is a picture of a Hadley's Sextant.

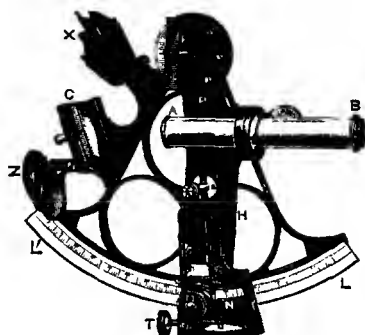


FIG. 72.*

AB is the telescope, C the fixed mirror corresponding to XY in Fig. 70, F the movable mirror corresponding to PQ.

PN is an arm by which the mirror F is turned; the amount of rotation being read off by means of the graduated arc LL', the vernier N, and the reading microscope M; fine adjustment is provided for by the tangent screw, T. H is a handle to hold the instrument by, X and Z are blackened glasses to interpose between the light and the telescope when either of the objects viewed is too bright, in order to make the intensities approximately equal.

The graduations of the arc are figured with twice their actual values. Thus an actual arc of 5° would be figured as 10° ; so that the reading of the arc is the actual angle subtended by the two objects.

* From an instrument supplied by Messrs Stanley, London.

TEST PAPERS.

§§ 1—5.

1.

1. Find the angle

(a) between the directions E.N.E. by $\frac{1}{4}$ E. and N.:

(b) between the hands of a clock at 5.42.

2. A man is facing E.S.E.: he turns his back to a north-westerly wind which is blowing: through what angle does he turn?

3. A man starts from home for a walk at 10 a.m.; after walking for 3 miles he turns off to the right at an angle of 30° , and proceeds for 25 minutes: he then turns to the right again and walks along a road at right angles for 2 miles, when he comes to a road that takes him straight home. Assuming that the roads are straight, and that he maintains an even speed of $3\frac{1}{2}$ miles an hour, draw a map to scale of his walk and deduce the time he reaches home.

4. Two rods AB, BC, each 1 ft. long, stand on a table with B vertically over the middle point of AC: the ends A and C are joined by a piece of thin elastic also a foot long. If, under the influence of the weights of the rods, B sinks through a distance of $2\frac{1}{2}$ inches, find the extension of the elastic AC.

§§ 1—6.

2.

1. Some boys out for a paperchase arrive at a hill-top; from which they see three churches that they recognize as those in three villages A, B, C marked on the map they carry. They judge that they are equally distant from all three, and find from the map that AB=10 miles, BC=12 miles, CA=17 miles. Shew by means of a figure drawn to scale (1 in. = 10 miles) how they determined their position, and give the distance of the hill-top from each village.

2. Draw an angle whose sine is $\frac{1}{6}$, and an angle whose cosine is $\frac{1}{3}$: construct an angle equal to the sum of these two angles and determine graphically its sine, cosine and tangent.

3. Find by geometrical constructions the values of

- (i) $\sin(2 \sin^{-1} \frac{3}{5})$;
- (ii) $\sin^{-1} \frac{5}{13} + \sin^{-1} \frac{12}{13}$;
- (iii) $\sin^2(\tan^{-1} \frac{8}{15}) + \cos^2(\tan^{-1} \frac{8}{15})$;
- (iv) $\sin(\cos^{-1} c)$, $\cos(\sin^{-1} s)$, $\tan(\sin^{-1} s)$, $\sin(\tan^{-1} t)$,
 $\cos(\tan^{-1} t)$, $\tan(\cos^{-1} c)$.

4. Shew that the sine and cosine of any acute angle are each less than unity, whereas no matter what numerical value is assigned to the tangent the acute angle having that number for its tangent can be found.

§§ 6—8.

3.

1. Given $\sin A = .263$, obtain *graphically* the values, correct to three decimal places, of $\cos A$, $\tan A$, $\cot A$, $\sec A$, $\operatorname{cosec} A$, taking 10 inches to represent the denominator of the fraction in each case and using a diagonal scale. Verify by decimal approximation that

- (a) $\cos^2 A + \sin^2 A = 1$,
- (b) $\sec^2 A - \tan^2 A = 1$,
- (c) $\operatorname{cosec}^2 A - \cot^2 A = 1$.

2. Solve the equation $2 \sin \theta + \cos \theta = 1$ by means of the identity $\sin^2 \theta + \cos^2 \theta = 1$, for $\sin \theta$ and $\cos \theta$: find the value of θ from the tables.

3. Given $a \cos \theta + b \sin \theta = c = b \cos \theta - a \sin \theta$, solve the equations for $\cos \theta$, $\sin \theta$; and by means of the identity $\cos^2 \theta + \sin^2 \theta = 1$ shew that $a^2 + b^2 = 2c^2$.

4. The following dimensions were taken from the end wall of a house:

| | |
|------------------|----------|
| height to eaves | 37 feet, |
| height to ridge | 51 feet, |
| breadth at eaves | 23 feet; |

if the two halves of the roof were of equal slope, find the angle of slope by calculation. Verify by a drawing to scale.

§§ 6—9.

4.

1. A tower stands on the top of a hill, whose side has a constant gradient from bottom to top. From the top of the tower the depressions of three points A, B, C which lie along the line of greatest slope through the foot of the tower are 34° , 36° , 40° . If $AB=BC=100$ ft., find the height of the top of the tower above the level of A by a diagram drawn to scale. [$1''=25$ ft. requires a $12'' \times 18''$ sheet of paper.]

2. If $\tan A + \sec A = 2$, shew that $\sin A = \frac{3}{5}$, A being an acute angle; also shew that $\cot A + \operatorname{cosec} A = 3$.

3. Solve question 2 by constructing on the same diagram, graphs of

$$y = \tan A + \sec A, \quad y = \sin A, \quad y = \cot A + \operatorname{cosec} A$$

from the tables, on as large a scale as convenient.

4. The sides of a rhombus are each 5 inches long, and the length of one diagonal is 9 inches, find the length of the other diagonal and the acute angle between two adjacent sides.

§§ 7—10.

5.

1. The elevations of the top and bottom of a flag-staff supposed to be 32 ft. high standing at the edge of a perpendicular cliff, taken from a certain station on the level sands at the foot of the cliff, are $48^\circ 50'$, $36^\circ 40'$ respectively, and the height of the cliff is calculated from these data. If the real height of the flag-staff is 31 ft. 8 in., what is the error in the height of the cliff?

2. Prove the following statements

$$(1) (\sin A + \cos A)^2 = 1 + 2 \sin A \cos A.$$

$$(2) \tan A + \cot A = \sec A \cdot \operatorname{cosec} A.$$

3. Solve the equations

$$(1) 2 \cos^2 \theta - 7 \cos \theta + 3, \quad \text{for } \cos \theta.$$

$$(2) 12 \tan^2 \theta - 13 \tan \theta + 3, \quad \text{for } \tan \theta,$$

and obtain a value of θ for each equation from the tables.

4. Plot the graphs of $y = \sin x$, $y = \cos x$, and obtain from them a graph of $y = \sin x + \cos x$ and $y = \cos x - \sin x$ for values of x between 0° and 90° .

5. Plot a graph of $s = 10t + 16t^2$; obtain the tangent of the angle of slope for successive values of t and plot a line to represent these values. What do the ordinates of points on this line represent?

§§ 7—12.

6.

1. Two adjacent sides of a parallelogram are 11, 17 inches respectively, and they include an angle of $37^\circ 12' 15''$. Find the lengths of the two diagonals and the area.

2. (i) If $\cot A + \operatorname{cosec} A = 5$, find $\cos A$;

(ii) if $\sec A = 7$, find the value of $\frac{\sin A + \cos A}{\tan A + \cot A}$.

3. Find the values of $\frac{n^2}{2} \cdot \sin \frac{360^\circ}{n}$, when n has the values 11, 21, 31, and $r = 10$ inches. [Note. These values are the areas of regular polygons of 11, 21, 31 sides respectively, inscribed in a circle whose radius is 10 inches.]

4. Find, by calculation from five-figure tables, the value of

$$\cos \frac{360^\circ}{7} + \cos \frac{720^\circ}{7} + \cos \frac{1080^\circ}{7} + \cos \frac{180^\circ}{7} \cos \frac{540^\circ}{7} \cos \frac{900^\circ}{7}.$$

5. The diameter of the base of a conical tent is 20 feet; the length of the narrow triangular strips of canvas, which are joined together to form the tent, is 17 ft. 6 in.; find the height of the tent.

§§ 7—13.

7.

1. A railway 200 miles long, which may be considered straight, crosses a country from sea to sea; the heights in feet above sea-

level which it reaches at the end of every 10 miles by the map are 500, 625, 670, 590, 542, 563, 700, 810, 920, 960, 978, 1020, 1033, 840, 706, 402, 383, 386, 394, sea-level. Find the tangent of the angle of slope for each section, as fractions with unit numerators.

2. Prove, by calculation from the tables, that

$$\sin^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{3\sqrt{11}} + \sin^{-1} \frac{3}{\sqrt{11}} = 90^\circ,$$

acute-angled values being taken for each \sin^{-1} .

3. From a ship sailing due N., two lighthouses are observed to bear N.E. and N.N.E. respectively: after the ship has sailed 20 miles the lighthouses are seen to be in a line due E. Find the distance between the lighthouses.

4. The ropes of a swing are 25 ft. long, the heights of the seat above the ground at its highest and lowest points are 14 ft. and 3 ft. respectively. What is the angle on each side of the vertical through which the swing oscillates?

5. ABC is a triangle, BD bisects the angle at A and meets BC in D. Prove $AB \cdot AC = AD^2 + BD \cdot DC$.

Hence find the angle BAC, if $AB=125$, $AC=80$, $AD=60$.

§§ 14—15.

8.

1. Prove the law for Multiplication of Logarithms directly from Definition (b) on p. 78, viz.:

$$a^{\log_a N} = N.$$

2. Find the values of the three fractions,

$$(i) \frac{1.732 \times 4.273}{8.634}; \quad (ii) \frac{1.732 \times 8.634}{4.273}; \quad (iii) \frac{4.273 \times 8.634}{1.732};$$

(a) by extracting from the five-figure tables on pages 88, 89, the *three-figure* logarithms of 17, 18, 42, 43, 86, 87, etc.:

(b) by extracting from the same tables, the *four-figure* logarithms of 173, 174, 427, 428, 863, 864, etc.:

(c) by using the *five-figure* logarithms given in the tables:

and find the percentage error by comparison with the answers, which have been calculated with *seven-figure* tables.

3. Shew that (i) $\log_a b \times \log_b c = \log_a c$,
(ii) $\log_a b \times \log_b a = 1$.

4. Find the values of

$$\log_2 0.5, \log_{0.5} 2, \log_8 \sqrt{2}, \log_{\sqrt{2}} \sqrt[3]{16}, \log_2 10.$$

5. Shew by means of the identity $\log_a b \times \log_b c = \log_a c$ that if a set of tables of logarithms to any base have been calculated, then a set to any other base can be obtained, by multiplying each logarithm in the first set by a constant factor.

§§ 14—17.

9.

1. Find by logarithms the value of

$$pv^\gamma,$$

when

$$p=726, \quad v=1.77, \quad \gamma=1.14.$$

2. Tables of logarithms are calculated to the base e , where $e=2.71828$ approximately: find the constant factor or **modulus** necessary to transform the tables to the base 10.

3. Calculate the area of the triangle ABC from the formula

$$\Delta = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C},$$

given $c=750$ ft., $A=46^\circ 27' 30''$, $C=75^\circ 16' 45''$.

4. Find the percentage error in the last result if the two angles, owing to a faulty instrument, are each $15''$ greater than the given observed values.

5. (a) In a triangle ABC shew that

$$(b+c) \sin \frac{A}{2} = a \sin \left(C + \frac{A}{2} \right),$$

and explain how to find the angles of a triangle, when an angle, the opposite side, and the sum of the other two sides are given.

(b) In the calculation you have to determine an angle from its sine; shew that, though this be the case, the data will give only **one** triangle.

(c) Given that one angle of a triangle is 110° , the opposite side is 5,000 units, and the sum of the other two sides 6,000 units, find the remaining angles to the nearest minute.

§§ 15—19.

10.

1. If d_1, d_2, d_3 are the distances of the centre of the inscribed circle from the angular points A, B, C of the triangle, prove

$$(i) \quad ad_1^2 + bd_2^2 + cd_3^2 = abc,$$

$$(ii) \quad \frac{d_1 d_2 d_3}{r} = \frac{2abc}{a+b+c}.$$

2. Given $\log 35 = a$, $\log 325 = b$, $\log 245 = c$, find

$$\log 5, \log 7, \log 13.$$

3. If the triangle circumscribing an isosceles triangle is equal to the escribed circle touching one of the equal sides, the triangle is right-angled.

4. Two sides of a triangle are 71 ft. and 25 ft. respectively, the contained angle is $69^\circ 32'$; solve the triangle and calculate the error in the third side due to a slight error, say of 2 inches, in the length of the smaller of the two given sides, i.e. side = 25 ft. 2 in. really.

5. (a) Explain the principle of the vernier. If a graduated arc reads angles to $10'$, how ought the vernier to be divided so that the arc may be made to give angles true to $10''$?

(b) Draw a part of a scale to represent inches and tenths of an inch from 19 to 21 inches; and draw a vernier in contact with the scale and so placed that the reading may be 20.26 in.

APPENDIX.

INTRODUCTION TO THEORETICAL TRIGONOMETRY.

§ 20. Circular measure.

The units employed for measuring angles, already given in § 2, are not as a rule used in Theoretical Trigonometry. The *theoretical* unit in all branches of mathematics is usually taken to be what is called an **absolute unit**: this being chosen, so that the measure of the quantity is represented by 1, when the measures of the fundamental quantities, such as mass, length and time, are also represented by 1.

Thus in Mechanics, the British Absolute Unit of Force is that Force which, if it acted on a mass of *one* pound for *one* second, would produce in it a velocity of *one* foot in *one* second. This is not the practical unit, which is about 32 times as great, namely, the weight of one pound.

Similarly, since an angle is the measure of the rotation of a vector, the *absolute unit* for angles is taken as the angle through which a vector of unit length has rotated round one fixed end, when its other end has described an arc of unit length. In other words it is the angle subtended at the centre of a circle whose radius is of unit length by an arc equal to the radius.

It cannot be concluded as an *immediate* deduction from this definition that the unit angle is the angle subtended at the centre of any circle by an arc equal to the radius; nor will this unit of angle be of any service unless it can be shewn that

(i) For different lengths of the vector, the arc described by the free end is proportional to the vector for the same angle.

(ii) For the same length of vector, the arc described by the free end is proportional to the angle.

In addition it is advisable that the relations between the absolute unit and the practical units, the right angle and the degree, should be known numbers; in other words it must be shewn that

(iii) The ratio of the circumference of a circle to its diameter is constant for all circles.

In modern text-books on Geometry, several practical methods for finding the approximate length of any curved line are given: but for circular arcs a *theoretical* method, which may be deduced from the following experiment, is generally adopted.

78. Set four stout pins firmly in a drawing board, fasten a piece of string* to A and pass it round B, C, and D and strain it tight enough to make the parts AB, BC, CD perfectly straight. Set a cylindrical piece of wood, such as are used for supports for physical or chemical apparatus (the cover of a round tin will do) in the position shewn in Fig. 73, arranging the pin B so that the parts BA, BC of the string just fail to touch the arc AC; that is BA and BC are tangents to the circular arc AC.

(i) Make a mark with a pen or pencil on the string where it touches the pin D.

* A tape measure substituted for the string is an improvement.

(ii) Take out the pin B, *pull* the free end of the string beyond D until the part between A and C fits tightly round the arc AC, and again make a mark on the string where it touches the pin D.

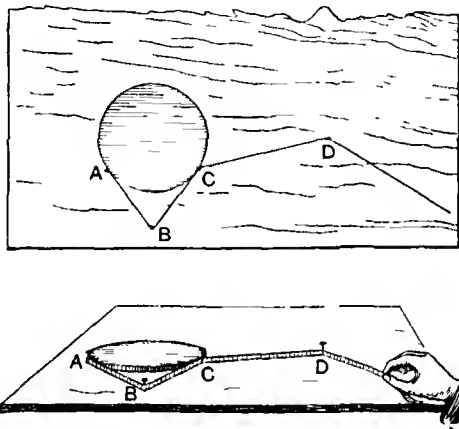


FIG. 73.

(iii) Take away the piece of wood or tin cover and again *pull* the string until the string between A and C is tight, i.e. straight, and again make a mark on the string where it touches the point D.

Since in Expt. 78, the free end of the string was lengthened by the *pull* at each change of the conditions of the experiment, (i), (ii) and (iii), the following conclusion is obvious:

The length of an arc is intermediate between the sum of the lengths of the tangents at its extremities and the length of the chord of the arc.

79. Take any arc AC, draw the tangents at A and C, join AC : measure AC and $AB + BC$. Then the length of arc AC lies between the lengths of AC and $AB + BC$.

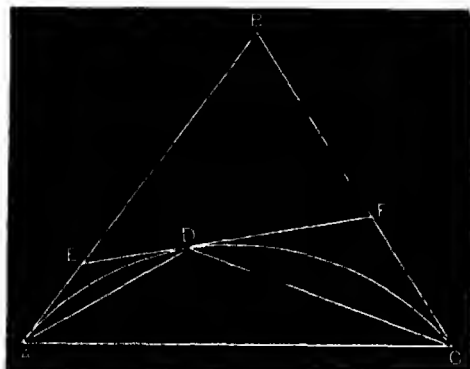


FIG. 74.

Take a point D on the arc and draw EDF to touch the circle at D and intersect AB, BC in E and F respectively ; join AD, DC : measure $AD + DC$ and $AE + EF + FC$. The length of the arc lies between these lengths. Also

$$AD + DC > AC \text{ and } AE + EF + FC < AB + BC.$$

Hence both the limits between which the length of the arc must lie are closer approximations than before.

Take two more points, one on the arc AD, the other on the arc DC, draw the corresponding tangents and chords, and obtain a third approximation, closer still, to the limits between which the length of the arc must lie.

Carry on the experiment as far as possible and tabulate the results as in the table below, which was obtained for an arc of $114\frac{1}{2}^\circ$ in a circle of $7\frac{1}{2}$ in. radius, the last result having been obtained by "stepping out" the arc with the divider points at $\frac{1}{4}$ -inch apart.

| | Sum of tangents | Sum of chords | Intermediate points |
|---|-----------------|---------------|---------------------|
| 1 | 23.50 | 12.65 | 0 |
| 2 | 16.75 | 14.30 | 1 |
| 3 | 15.45 | 14.85 | 3 |
| 4 | 15.05 | 14.95 | 8 |
| | — | 15.00 | 59 |

The results tabulated above may be shewn graphically, when rate of effect produced per extra intermediate point is seen at a glance.

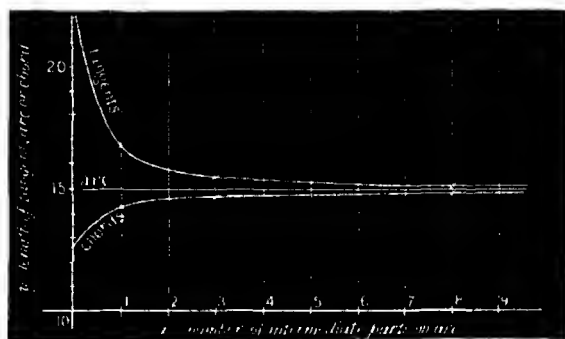


FIG. 75.

For the purpose of the experiments which follow, the student may *assume* that the length of an arc is very approximately given by carefully stepping it out with the points of the dividers at $\frac{1}{16}$ -inch apart, the error in defect being less than .1 per cent. of the total length of the arc, for any circle whose radius is greater than one inch.

80. Draw any acute angle BAC : with centre A draw arcs P_1Q_1, P_2Q_2, \dots . Measure the several arcs (PQ), the corresponding radii (AP) and obtain as a decimal the ratio $\left(\frac{PQ}{AP}\right)$, and exhibit the result in tabular form.

Repeat the experiment for an obtuse angle.

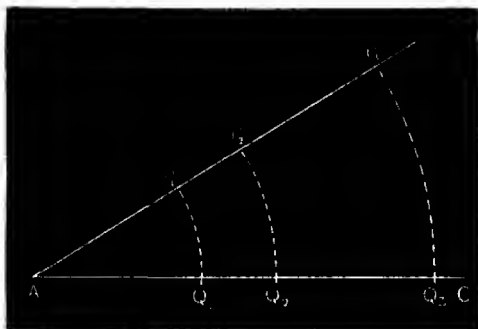


FIG. 76.

If the measurement of the arcs, by “stepping out” with the divider points at a distance of $\frac{1}{10}$ -inch apart, is carefully performed, any excess beyond $\frac{1}{10}$ -inch, being either estimated by eye or measured on a diagonal scale reading to $\frac{1}{100}$ -inch, the results in the column for the ratio should be the same to the second or third place of decimals. Deduce the following theorem:

For a given angle at the centre of a system of concentric circles, the subtending arcs are proportional to the radii.

81. Draw a circle of any conveniently large radius. Draw several radii $OXOA_1, OA_2, OA_3, \dots$. Measure the arcs XA_1, XA_2, XA_3, \dots , and the $\angle s$ $XOA_1, XOA_2, XOA_3, \dots$, with a protractor, and deduce the following theorem:

In any given circle, angles at the centre of the circle vary as the arcs on which they stand.

The theorem deduced in Expt. 81 follows naturally from the theorem, proved on page 5, that equal arcs of a circle subtended equal angles at the centre.

For, let O be the centre of a circle, and AOB , BOC , two angles at the centre standing on the arcs AB , BC ; let XY be an arc, which is contained p times in AB and q times in BC .

Divide AB into p equal arcs, and BC into q equal arcs each equal to XY .

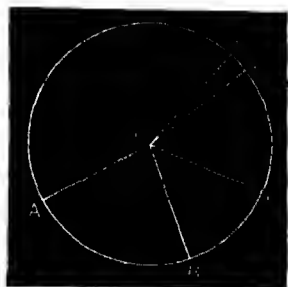


FIG. 77.

Then each of these arcs will subtend at O a number of equal angles, each equal to XOY , of which AOB will contain p , and BOC will contain q .

$$\therefore \frac{\angle AOB}{\angle BOC} = \frac{p}{q} = \frac{AB}{BC}.$$

If s , a , θ , be the measures of the arc, the radius, and the angle subtended by the arc at the centre, of any circle, it has been shewn that

- (i) s varies as a , when θ is constant;
- (ii) s varies as θ , when a is constant.

Hence, by the theory of variation (see Algebra)

s varies as $a\theta$ when both a and θ vary.

Or symbolically

$$s = c_1 \cdot a \quad (\theta = \text{const.}),$$

$$s = c_2 \cdot \theta \quad (a = \text{const.}),$$

$$\therefore s = c \cdot a \cdot \theta,$$

where c_1 , c_2 , c are constants.

If, then, we choose as unit angle, that angle which is subtended at the centre of a circle by an arc equal to the radius, we have in the formula $s = c.a.\theta$,

$$s = a \text{ when } \theta = 1,$$

$$\therefore c = 1,$$

and the formula becomes $s = a\theta$; i.e., the arc measures the angle on a scale whose unit equals the radius.

82. Draw circles of radii, $3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 5$ inches respectively; measure the circumferences by "stepping out" with the points of the dividers at $\frac{1}{10}$ -inch apart; obtain the ratio $\frac{\text{circumference}}{\text{diameter}}$, and tabulate the results.

Deduce that in any circle, the ratio of its circumference to its diameter is constant.

If the measurement is carefully done, the results in the last column should agree to at least two places of decimals. The correct answer true to two places of decimals is 3.14. This ratio or abstract number, for which the letter π is used, may also be determined in a manner similar to that in Expt. 79, by using the fact that the circumference is intermediate between the perimeters of regular polygons inscribed in and circumscribed about the circle.

83. Let A, B, C..., D, E, F, G... be the angular points of regular polygons of n sides, inscribed in and circumscribed about a given circle, whose centre is O, and whose radius is r .

Join OA, OB, OC, OE, OF: shew that

$$(1) \text{ perimeter of inscribed polygon} = n \cdot AB = 2nr \cdot \sin \frac{360^\circ}{2n},$$

$$(2) \quad \text{,, circumscribed ,,} = n \cdot EF = 2nr \cdot \tan \frac{360^\circ}{2n}.$$



FIG. 78.

Using the formulae obtained for the perimeters of inscribed and circumscribed regular polygons of n sides, the following table may be constructed from tables of natural sines and tangents, for a circle whose radius is $\frac{1}{2}$ -inch and whose circumference is therefore π inches.

| n | $2nr \sin \frac{360^\circ}{n}$ | $2nr \tan \frac{360^\circ}{2n}$ | n | $2nr \sin \frac{360^\circ}{n}$ | $2nr \tan \frac{360^\circ}{2n}$ |
|-----|--------------------------------|---------------------------------|-----|--------------------------------|---------------------------------|
| 5 | 2.93895 | 3.63270 | 36 | 3.13776 | 3.15964 |
| 6 | 3.00000 | 3.46410 | 40 | 3.13840 | 3.14800 |
| 8 | 3.06144 | 3.31368 | 60 | 3.14040 | 3.14460 |
| 10 | 3.09020 | 3.24920 | 72 | 3.14064 | 3.14352 |
| 12 | 3.10584 | 3.21540 | 90 | 3.14100 | 3.14280 |
| 15 | 3.11865 | 3.18840 | 120 | 3.1412280 | 3.1423080 |
| 18 | 3.12570 | 3.17374 | 180 | 3.1414320 | 3.1419180 |
| 20 | 3.12860 | 3.16760 | 360 | 3.1415400 | 3.1416840 |
| 24 | 3.13272 | 3.15960 | 720 | 3.1415760 | 3.1416480 |

The results in the foregoing table for the perimeters of the polygons are correct to within .0001 for $n=5$ to $n=20$, to within .0005 for $n=20$ to $n=90$, and to within .00003 for $n=90$ to $n=720$, the last four results being obtained from seven-figure tables. By considering the differences in these last four it is evident that the perimeter of the circumscribed polygon shows a rate of decrease which is greater than the rate of increase of the perimeter of the inscribed polygon. Hence the circumference of the circle is about 3.1416.

It has been deduced from the results obtained in Expt. 82 that the ratio of the circumference of any circle to its diameter is constant for all circles. This can be theoretically proved as follows.

N.B. It is assumed that, if AB, AC are two tangents to a circle, then $AB + AC > \text{arc } BC > \text{chord } BC$. [Cp. p. 147.]

Suppose LA, AM are two sides of a regular polygon of n sides circumscribed about the circle, touching the circle at their middle points B, C : then BC is a side of a regular polygon of n sides inscribed in the circle.

Bisect the arc BC at D and draw EDF tangent to the circle at D : then it will be seen, by symmetry, that EF is the side of a regular polygon of $2n$ sides circumscribed to the circle, and BD is a side of a regular polygon of $2n$ sides inscribed in the circle.

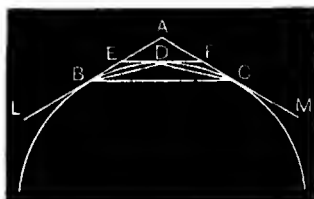


FIG. 79.

Again, since the sides of a circumscribed polygon are bisected at their points of contact with the circle,

$$\begin{array}{ll}
 \text{the perimeter of the circumscribed } n\text{-gon} &= n(AB + AC), \\
 \text{'' '' '' } 2n\text{-gon} &= 2n \cdot EF \\
 &= n[BE + EF + FC], \\
 \text{the perimeter of the inscribed } n\text{-gon} &= n \cdot BC, \\
 \text{'' '' '' } 2n\text{-gon} &= 2n \cdot BD \\
 &= n \cdot [BD + DC].
 \end{array}$$

But, since two sides of a triangle are greater than the third, it is clear that

$$BD + DC > BC,$$

$$\text{and } AB + AC > BE + EF + FC.$$

Hence, if the number of sides be doubled, the perimeters of regular polygons circumscribing the circle are diminished, and the perimeters of regular polygons inscribed in the circle are increased.

But it follows, from the assumption made above, that if ${}_nP_c, S, {}_nP_i$ are the perimeter of a circumscribed n -gon, the circumference of the circle, the perimeter of an inscribed n -gon respectively,

$${}_nP_c > {}_{2n}P_c > S > {}_{2n}P_i > {}_nP_i.$$

Hence, if $S = nP_e - d_e$, or $S = nP_i + d_i$,

d_e, d_i are diminished indefinitely as the number of sides become $2n, 4n, 8n$, etc., and can be made smaller than any assignable quantity by taking the number of sides large enough. Hence the circumference of a circle may be considered as equal to the perimeter of a circumscribing (or of an inscribed) polygon, the number of whose sides is infinitely great.

Now, suppose $ABCD \dots, A'B'C'D' \dots$ are any two circles, placed so that their centres coincide at O .

Let $A, B, C, D \dots$ be the vertices of a regular polygon of n sides inscribed in the greater circle. Join $OA, OB, OC, OD \dots$, cutting the smaller circle in $A', B', C', D' \dots$ respectively.

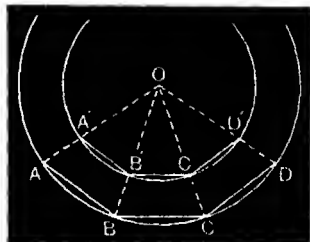


FIG. 80.

Then $A', B', C', D' \dots$ will be the vertices of a similar and similarly situated polygon; i.e. a regular polygon of n sides, each side being respectively parallel to a side of the polygon $ABCD \dots$.

Hence, $\therefore \Delta s OBC, OB'C'$ are equiangular,

$$\therefore \frac{BC}{OB} = \frac{B'C'}{OB'},$$

$$\therefore \frac{\text{perimeter of } ABCD \dots}{OB} = \frac{\text{perimeter of } A'B'C'D' \dots}{OB'};$$

this is true, no matter how great the number of sides of the polygons may be.

It is therefore true when the number is infinitely great; that is, it is true when the perimeters are equal to the circumferences of the circles respectively,

$$\frac{\text{circumference of } \odot ABCD}{OB} = \frac{\text{circumference of } \odot A'B'C'D'}{OB'}.$$

The true value of π has been worked out to several hundred decimal places by methods of more advanced Trigonometry, and it has been proved that it is a never-ending, non-recurring decimal, i.e. an **incommensurable** number. As far as the tenth decimal place,

$$\pi = 3.1415926536.$$

Several approximations are used:—

- (1) $2\frac{2}{7}$ or $3\frac{1}{7} = 3.142857$ for rough calculations;
- (2) 3.1416 (divisible by 3, 7, 8, 11, 17), and
- (3) $3\frac{55}{113} = 3.1415929\dots$ for more exact work.

The measure of an angle in absolute units is called the **circular measure of the angle**. To avoid speaking of “an angle whose circular measure is n ,” the suggestive name **radian** was invented for the unit: thus we *may* speak of “an angle of n radians,” instead of “an angle whose circular measure is n ,” just as we speak of “an angle of x degrees.”

It follows at once from what has already been proved in this section that the circular measure of an angle varies as the number of degrees in the angle, and as soon as the angle under consideration has been expressed in degrees, and a fraction of a degree, its circular measure can be deduced by simple proportion, and *vice versa*, from the fundamental relation between the two measures of an angle of two right angles.

The circular measure of two right angles, by using the formula $s = a\theta$, is found to be $\frac{\text{semi-circumference}}{\text{radius}}$; that is, π .

$$\therefore 180 \text{ degrees} \equiv \pi \text{ radians.}$$

84. Find the circular measure of an angle of $43^{\circ} 20' 30''$.

The circular measure of 180° is π ,

$$\therefore \quad \quad \quad \quad 1^{\circ} \text{ is } \frac{\pi}{180},$$

$$\begin{aligned} \therefore \quad \quad \quad \quad 43^{\circ} 20' 30'' &\text{ is } \frac{\pi}{180} \times 43 \frac{41}{120} \\ &= \frac{355 \times 520 \cdot 1}{113 \times 180 \times 120} \\ &= \cdot 75645. \end{aligned}$$

85. Find the number of degrees, minutes, and seconds (i.e. the sexagesimal measure) of the unit of circular measure.

Since π radians contain 180° ,

$$\begin{aligned} \therefore \text{unit of circular measure} &= \frac{180^{\circ}}{\pi} \\ &= 180^{\circ} \times \frac{113}{355} \\ &= 57^{\circ} 17' 45'' \text{ nearly.} \end{aligned}$$

1. Find the number of degrees, minutes and seconds in an angle whose circular measure is 2.5.

2. Find the circular measure of an angle of $36^{\circ} 15' 12''$.

3. If the radius of a circle is 10,000 ft., find to the nearest foot the length of an arc which subtends an angle of one minute at the centre.

4. The distance between successive graduations on the outer rim of a graduated circle is one-fiftieth of an inch, and this interval subtends an angle of one minute at the centre of the circle; find the diameter of the circle to the nearest inch.

5. A man runs round a circular track at 12 miles an hour. The circular measure of the angle through which his body has turned is 1.5 at the end of a minute; find the diameter of the track.

6. How long does it take the hour-hand of a watch to rotate through the angle whose circular measure is $\frac{\pi}{15}$?

The area of a circle can be obtained by a method similar to that by which the length of the circumference was found in Expt. 83. For the area is intermediate between those of the inscribed and circumscribed regular polygons of n sides.

From Fig. 78 on p. 152, it follows that

$$(1) \text{ Area of inscribed polygon} = \frac{1}{2}nr^2 \sin \frac{360^\circ}{n}.$$

$$(2) \quad \text{,, circumscribed ,,} = nr^2 \tan \frac{360^\circ}{2n}.$$

If we calculate the values of $\frac{1}{2}n \sin \frac{360^\circ}{n}$ and $n \tan \frac{360^\circ}{2n}$, we obtain

| n | $\frac{1}{2}n \sin \frac{360^\circ}{n}$ | $n \tan \frac{360^\circ}{2n}$ | n | $\frac{1}{2}n \sin \frac{360^\circ}{n}$ | $n \tan \frac{360^\circ}{2n}$ |
|-----|---|-------------------------------|-----|---|-------------------------------|
| 5 | 2.37765 | 3.63270 | 36 | 3.12570 | 3.15364 |
| 6 | 2.59806 | 3.46410 | 40 | 3.12860 | 3.14800 |
| 8 | 2.82844 | 3.31368 | 60 | 3.13590 | 3.14460 |
| 10 | 2.93895 | 3.24920 | 72 | 3.13776 | 3.14352 |
| 12 | 3.00000 | 3.21540 | 90 | 3.13920 | 3.14280 |
| 15 | 3.05055 | 3.18840 | 120 | 3.1401600 | 3.1423080 |
| 18 | 3.07818 | 3.17374 | 180 | 3.1409550 | 3.1418180 |
| 20 | 3.09020 | 3.16760 | 360 | 3.1414320 | 3.1416840 |
| 24 | 3.10584 | 3.15960 | 720 | 3.1415400 | 3.1416480 |

with the same limits of accuracy as in the table for perimeters on p. 153: from which it is easy to deduce that

the area of a circle of radius $r = \pi r^2$.

Again, \therefore area of circle $= \pi r^2 = r \cdot \pi r$,
and semi-circumference $= \pi r$ by Expt. 83,

\therefore the area of a circle is equal to that of a rectangle whose adjacent sides are equal to the radius and the semi-circumference respectively.

This can be shewn practically by a method of dissection, as follows:

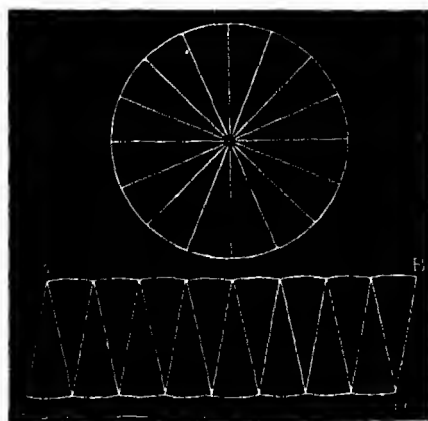


FIG. 81.

Draw a circle of any radius on paper and cut it out: divide it into a number of equal parts as in Fig. 81. Rearrange the parts, as in the lower diagram ABCD. Note that if the number of pieces is increased the broken lines AB, CD become more and more nearly straight: and ultimately, when the number of pieces is very great, the curves disappear and the lines AB, CD become straight lines equal in length to the semi-circumference of the circle.

Moreover the angles at D and B become right angles.

Hence the figure becomes ultimately a rectangle whose adjacent sides are respectively equal to the radius and semi-circumference of the given circle.

EXERCISES.

1. The driving wheel of a locomotive is 7 ft. 6 in. high: how many revolutions per minute does it make when the speed of the train is 60 miles per hour, supposing that there is no slipping between the wheel and the rail?

2. Express in circular measure the interior angles of regular polygons of 5, 6, 8, 10 sides.

3. Two pulleys of radii 3 in. and 7 in., are set on short parallel axes projecting at right angles from a vertical wall at a distance 12 in. apart. Find, to the nearest inch, the length of the belt which will go round these pulleys, so that they rotate in the same direction, i.e. the belt does not cross between the pulleys.

4. Find the length of the belt if the pulleys in Ex. 3 have to rotate in opposite directions, i.e. the belt crosses between the pulleys.

5. An equilateral triangle is inscribed in a circle, and a regular hexagon is circumscribed to the same circle: find the ratio of their areas.

6. The areas of regular polygons, of the same number of sides, inscribed in and circumscribed to a circle, are in the ratio 3 : 4. Find the number of sides.

7. Give a formula to find the sectional area of a cylinder, the circumference only being given. What must be the girth of a water pipe, made of iron $\frac{1}{2}$ -inch thick, to give an effective sectional water area of one square foot?

8. A florist exhibiting a group of plants at a show wishes to arrange them with the tallest in the middle in the form of a circular cone: each competitor is allowed 50 sq. ft. of floor for his collection: find the length of the string which he must use as a radius to mark out the circle on the floor.

9. A circle of 10" radius is divided into three equal parts by the circumferences of two concentric circles: find the radii of the two circles.

10. How much will it cost to turf a circular lawn, whose diameter is 35 yds., at 6d. a sq. yd.?

§ 21. Use of negative sign in Graphs.

In Geometry and Graphical Algebra it is very often necessary to consider *direction* as well as *magnitude* in determining the length of a straight line. Thus, if A and B are two points, the distance of A and B (say, two miles *North-West*) is considered to be different from the distance of B from A (two miles *South-East*). The most convenient method of indicating opposite directions along a straight line is by the use of the algebraical signs + and -; that direction which is most suitable in each particular case being *chosen* for the positive direction of measurement. Thus, if we consider South-East to North-West as positive, the distance of A from B, called *the length of BA* is + 2 miles, and the distance of B from A, *the length of AB*, is - 2 miles.

Similarly, the two possible directions of rotation in describing angles are distinguished from one another by choosing one direction as the positive direction, the other direction being considered negative. The usual convention is to reckon as **positive** angles described by an **anti-clockwise** rotation, i.e. in a direction contrary to that in which the hands of a watch move.

When account is thus taken of the direction of measurement, or **sense**, of a line or angle, some **zero position** must be chosen. This, for measurement of lines, is called the **origin** or **pole**; and, for measurement of angles, the **initial line**. The origin and initial line are chosen to suit the needs of each particular case.

86. Take a point O as origin on any straight line and mark the points A, B, C, D, E so that $OA=3$, $AB=5$, $BC=-6$, $CD=2$, $DE=-9$. Write down the lengths of BA, BD, EO; and verify that

$$(1) \quad OA+AB+BC+CD+DE=3+5-6+2-9=-5=OE.$$

$$(2) \quad AB+BC+CA=0.$$

To fix the position of a point on a certain surface, the usual method is to find two lines, *which can easily be determined*, intersecting in the point. Thus, (1) places on the earth's surface are fixed by the latitude and longitude; (2) the usual method of piloting a ship along a difficult channel, such as the entrance to Liverpool, is to keep two landmarks, beacons, or buoys in a line until two others are in a line, the ship being then at the point of intersection of the two lines, which indicates the point at which the course has to be changed; (3) buried treasure can be hidden and its position found again by the intersection of two circles whose radii are known; for this two origins or poles are required, such as the distances from a tree and a telegraph post, and also directions for distinguishing the particular tree and post; (4) the distance and bearing of one point from another give a circle of known radius and a straight line making a known angle with a fixed direction, say North. If, however, two circles are used as in (3), they intersect in two points and it is necessary to have some other thing to determine which of the two points is the right one. With a circle and a straight line drawn from the centre, as in (4), although they intersect in two points, the point can be at once determined if we have some convention for an angle similar to that which distinguishes E.N.E. from E.S.E., which both make angles of $22\frac{1}{2}^\circ$ with E., i.e., the idea of positive and negative angles: this, one of the most important methods of fixing a point, will be again referred to in § 23.

Another method, the most generally used, is that in which the position of the point is determined as the intersection of two straight lines parallel to two fixed straight lines, the distance of each line from the initial line, to which it is parallel, being measured *parallel to the other line*. This is the method of "squared-paper" work, though it is not necessary that the initial lines should be at right angles.

87. Draw any two straight lines OX, OY ; produce them backwards through O to X', Y' .

For distances along straight lines parallel to $X'X$ or OX , consider those as positive that are drawn in the same sense as OX , and as negative, those in the opposite sense: for distances along straight lines parallel to $Y'Y$ or OY , consider those as positive which are drawn in the same sense as OY and as negative, those in the opposite sense mark the points A, B, C, D, E, F, G, H from the following measurements:



FIG. 82.

| | A | B | C | D | E | F | G | H |
|--|---|---|---|----|----|----|----|---|
| Distance from OY parallel to OX | 3 | 2 | 4 | -5 | 0 | -3 | 4 | 0 |
| Distance from OX parallel to OY | 2 | 3 | 0 | 4 | -6 | -2 | -3 | 0 |

The lines $X'OX, Y'OY$ are called **axes of coordinates** and the point O the **origin**.

If RP and SP denote the distances, measured in the senses of OX, OY respectively, which fix the position of the point P , RP and SP are called the ***Cartesian Coordinates of the point P** , or simply, the **coordinates of P** .

* After Des Cartes, the great mathematician.

In finding a point P , however, it is in general more convenient, since OS is equal to RP , to mark off a distance OS *along* OX and then the distance SP *parallel to* OY , or OR , OS may both be marked off along the axes and the parallelogram $SORP$ be completed.

If the straight lines $X'OX$, $Y'OY$ are taken at right angles, the parallelogram becomes a rectangle; the axes are then called **rectangular axes**, SP the **ordinate**, and OS the **abscissa**. When these lengths are given it is most important to know the signs of the lengths and also which is mentioned first.

88. A small chart is discovered containing a rough sketch, with directions, of the position of buried treasure.

The place is easily recognised by the discoverer of the chart by the four trees which are each 30 feet from the intersection of the hedges.

If the angle between the hedges is 65° , draw a diagram showing the *eight different positions* where the treasure might lie.



FIG. 88.

The abscissa is generally referred to as the x -coordinate, and the ordinate as the y -coordinate of a point; and if it is agreed that the x -coordinate shall always be mentioned first, a point P whose coordinates are

$$\left. \begin{array}{l} x = 3 \\ y = 4 \end{array} \right\},$$

can be referred to as the point " $P[3, 4]$ " or, simply, as the point " $(3, 4)$."

89. Take rectangular axes and mark the position of the points

$$A[3, 4], B[4, -5], C[-5, -1], D[-1, 3]$$

using $\frac{1}{2}$ in. as the unit.

Calculate, by Pythagoras' Theorem, the length of AB, BC, CD, DA, AC; verify by direct measurement.

It should be carefully observed what

$$\left. \begin{array}{l} x = 3 \\ y = 4 \end{array} \right\} ,$$

as the coordinates of a point P, really mean. If a series of points, for which the distance from the y -axis for each is equal to +3, are plotted, these points all lie on a *straight line* parallel to the y -axis, and $x=3$ is called the *equation of the line*. Similarly $y=4$ is the equation of a line parallel to the x -axis, the distance of each point on it from the x -axis being equal to +4. The point P is the intersection of these lines, the only point for which $x=3$ and $y=4$.

Similarly $x=y$ is the equation of the straight line, each point of which is equidistant from the two axes. This is the straight line, *produced both ways through O*, which bisects the angle XOY.

90. Plot the line whose equation is $x=-y$, i.e. $x+y=0$.

§ 22. Angles of any magnitude.

Whilst in surveying and for most, but not all, practical purposes it is only necessary to consider angles less than two right angles, as given by the Euclidean definition; in theoretical trigonometry no such limitation is made to the magnitude of an angle. An angle is defined as the *result of a certain operation*, viz. rotation.

DEF. When a straight line rotates in a plane about some point in its length from any one position to any other position it is said to have described an angle: the amount of rotation measures the angle. [Cf. p. 152.]

With this definition, given only the initial and final positions of the rotating line, it is impossible to fix the absolute magnitude of the trigonometrical angle. For it is not known (i) how many complete revolutions have been made, nor (ii) in which direction the rotation has taken place.

Thus even supposing, in Fig. 84, that no complete revolution has been made, the amount of rotation round O by which a line may be brought from the position OA to the position OB is obviously different for the two directions denoted by the arrow heads. If the circular measure of the positive angle represented by an arrow in the figure is θ , and that of the negative angle is $-\phi$, then

$$\theta + \phi = 2\pi,$$

since 2π is the circular measure of 4 rt. angles.



FIG. 84.

Now all the positive angles that could have been described with the initial and final positions of the rotating line as in the figure are contained in the series

$$\theta, \theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \dots;$$

and all the negative angles are included in the series

$$-\phi, -\phi - 2\pi, -\phi - 4\pi, -\phi - 6\pi, \dots,$$

i.e. $\theta - 2\pi, \theta - 4\pi, \theta - 6\pi, \dots;$

Hence all the angles which are determined by the inclination of the two straight lines OA, OB are included in the formula

$$\theta + 2n\pi,$$

where n is any integer positive or negative.

The angle between any two non-intersecting straight lines is equal to the angle between any two straight lines drawn parallel to and in the same sense as the given straight lines.

In practice, one of the given straight lines is usually considered as the initial line, and a straight line is drawn intersecting it which is parallel to and in the same sense as the second of the given straight lines.

Thus in Fig. 85, we consider OA as the initial line and draw OD parallel to and in the same sense as PQ.

Then the angle between PQ and OA is the angle AOD, either the big positive angle, or the small negative angle—marked in



FIG. 85.

Fig. 85 with arrows—or these angles increased or diminished by some multiple of four right angles.

It is important to remember the exact way in which such a line as OD is drawn :—

Through the point O (the point of departure in OA), OD is drawn parallel to and in the same sense as PQ.

EXERCISES.

1. Mark with arrows the positive and negative angles less than four right angles between the straight lines (Fig. 85).

(i) PQ and AO,

(ii) QP and OA,

(iii) QP and AO,

using OA as the initial line.

2. Repeat Ex. 1 using PQ as the initial line.

3. Draw figures showing angles of :—

(i) 270° , (ii) -235° , (iii) 1000° , (iv) -625° ;

give the smallest positive and the smallest negative angles for each case, which the positions of the arms of the angles might determine.

4. Draw any irregular pentagon on a large scale, cut it out and tear off the corners; arrange the angles round a point so as to shew that their sum is six right angles.

5. Repeat Ex. 4 for the case of irregular or regular rectilinear figures of 6, 7, 8, 10 sides respectively, and verify the formula: The sum of all the interior angles of a polygon of n sides is equal to $2n - 4$ right angles.

§ 23. Trigonometrical ratios for angles of any magnitude.

Let a straight line OP , rotate round O through an angle whose circular measure is θ . Let its initial position coincide with OX , and its final position with OP in one of the four diagrams in Fig. 86, where a positive value

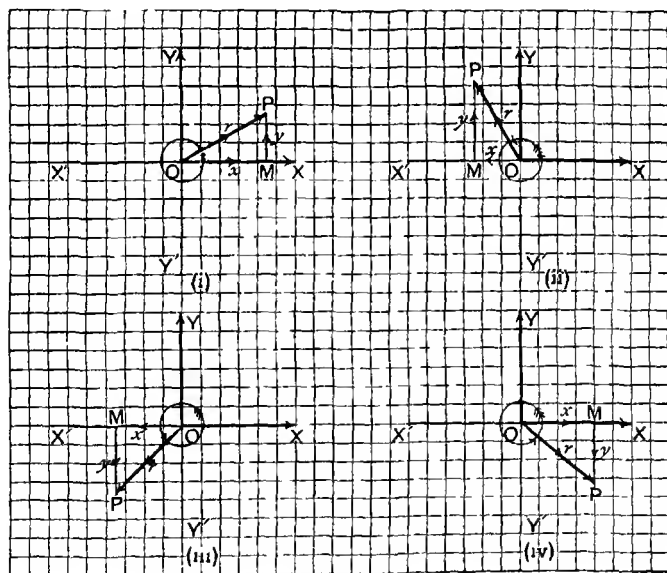


FIG. 86.

has been indicated for θ , by the arrows, although what follows is also true if the position OP had been reached by a negative rotation.

If r is the length of OP , it was shewn in § 21 that r and θ determined the position of the point P : r and θ are called the **polar coordinates** of P referred to O as pole and OX as the initial line. Let OY make with OX an angle equal to $+90^\circ$ or $+\frac{\pi}{2}$ (the circular measure of 90°):

let x and y be the cartesian coordinates of P . Then the general values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ may be thus defined.

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad \tan \theta = \frac{y}{x},$$

where x and y are taken positive when measured in the same sense as OX and OY respectively, and r , which is really $\pm \sqrt{x^2 + y^2}$, is taken as positive when measured outwards from the origin along the radius vector.

These ratios may also be defined in terms of the sides of the triangle OPM in Fig. 86, as

$$\sin \theta = \frac{MP}{OP}, \quad \cos \theta = \frac{OM}{OP}, \quad \tan \theta = \frac{MP}{OM},$$

if the order of the letters, determining the sign of the line, is carefully noted. In the figures arrow-heads are placed on the ends of the lines OX, OY to shew the directions of positive measurements for x and y respectively, and the arrow-heads on the sides are shewn pointing from O to M , and from M to P . The agreement (or otherwise) in direction between these latter and those on the lines OX, OY denoting the directions of positive measurement at once determine the signs of the ratios.

91. Find the value of $\tan 150^\circ$ or $\tan \frac{5\pi}{6}$.

It is evident from Fig. 87, that MP is positive and OM is negative;

$$\therefore \tan \frac{5\pi}{6} = \frac{MP}{OM} = \text{a negative quantity.}$$

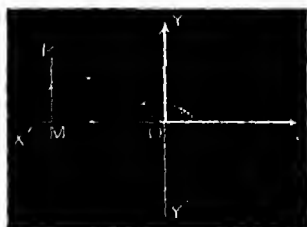


FIG. 87.

Again the $\triangle OPM$ is half an equilateral triangle whose sides are equal to OP ; hence, if $OP=r$, the lengths of the straight lines MP , OM (without regard to sign) are $\frac{r}{2}$, $\frac{\sqrt{3}r}{2}$ respectively.

$$\therefore \text{arithmetical value of } \frac{MP}{OM} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.57735.$$

$$\therefore \tan \frac{5\pi}{6} = -0.57735.$$

92. Find in a similar manner the values of

$$(1) \sin 240^\circ, \quad (2) \tan \frac{5\pi}{4}, \quad (3) \cos (-135^\circ),$$

$$(4) \cos (-1110^\circ), \quad (5) \cos \pi, \quad (6) \tan \frac{\pi}{2}.$$

§ 24. Graphs of the Trigonometrical Ratios for angles of any magnitude.

It is plain from § 23 that the sign and magnitude of the trigonometrical ratios of an angle are always the same when the rotating line occupies the same position, no matter how many complete revolutions it has made, in either the positive or negative direction of rotation: for the ratios only depend on the final *position* of the rotating line. Also no matter what was the initial magnitude of an angle A , when the rotating line has made one complete revolution either positively or negatively, and again reaches the position which determined the angle A , each of the ratios has passed through every possible value both as regards sign and magnitude: and if the rotation still goes on each of the ratios will *repeat* the same set of all possible values, in exactly the same order, for every complete revolution.

When one quantity depends on a second quantity in such a way that as *successive* values are given to the second, the first repeats a certain set of all possible values in exactly the same order, the first quantity is said to be a *periodic function* of the second.

Thus $\sin A$, $\cos A$, $\tan A$ are periodic functions of A .

Again, since the other ratios are *defined* by the relations

$$(1) \quad \operatorname{cosec} A \sin A = 1,$$

$$(2) \quad \sec A \cos A = 1,$$

$$(3) \quad \cot A \tan A = 1,$$

it follows that $\operatorname{cosec} A$, $\sec A$, $\cot A$ are also periodic functions of A . It should also be observed that the *cosec* and *sin*, the *sec* and *cos*, the *cot* and *tan* have their signs respectively equal for all angles.

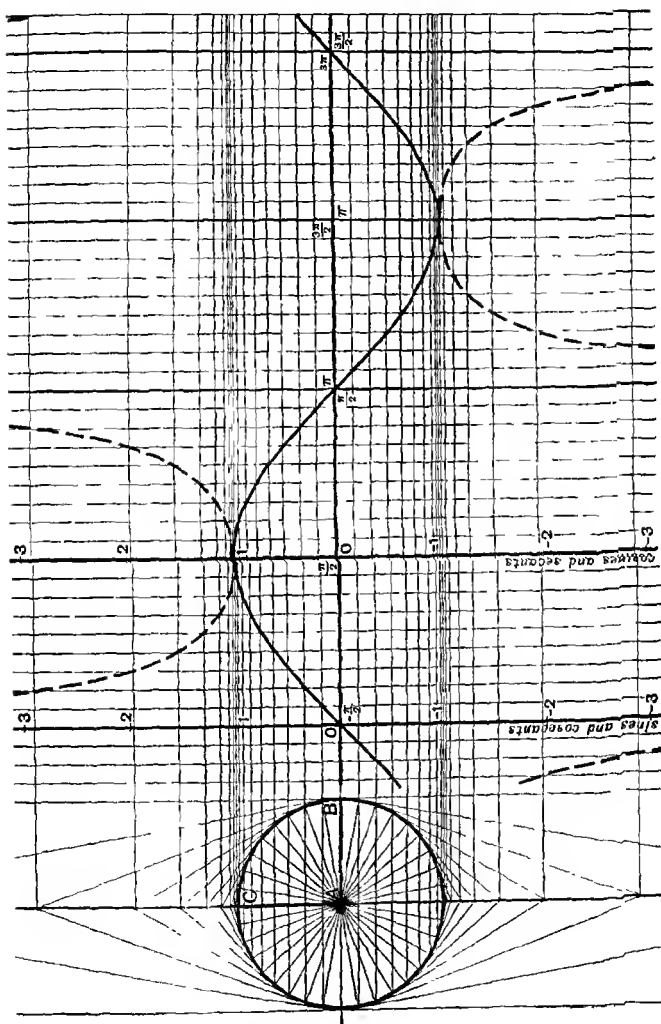


FIG. 88

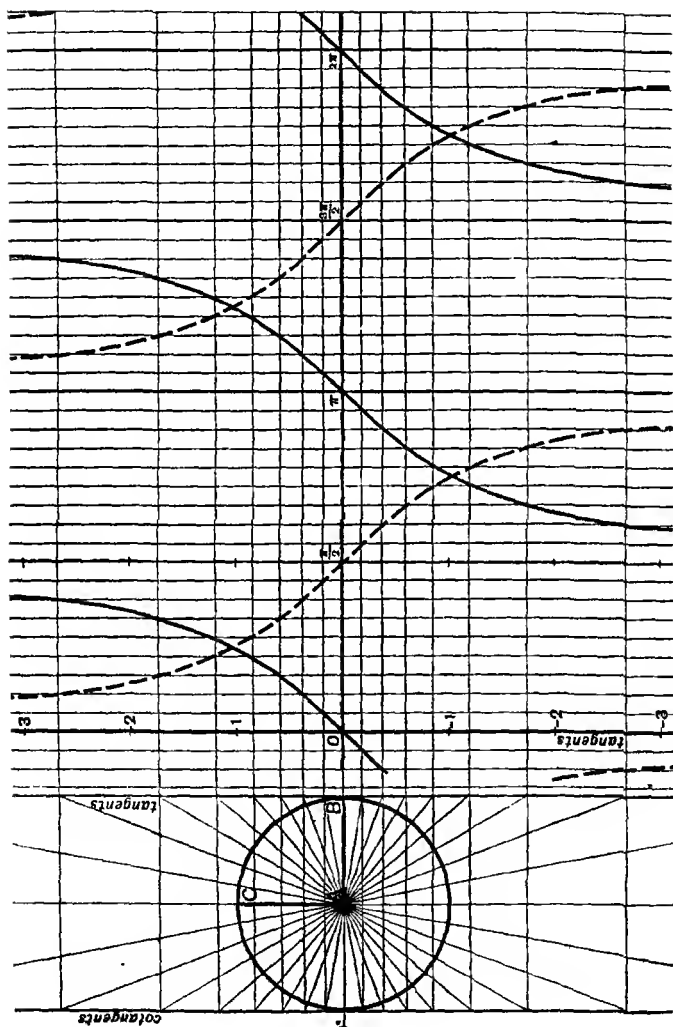


FIG. 89.

The variation and periodicity can best be shewn from a graph, *geometrically drawn*, i.e. not plotted from tables. Referring to the diagram on p. 45, the method used in drawing the graphs on pp. 173, 174 is obvious for the *sine* and *cosecant*: for the *cosine* and *secant*, in order to project horizontally the values of these ratios the diagram must be given a twist of $+\frac{\pi}{2}$; or, in other words, for the complementary ratios AC must be chosen as the initial line instead of AB: for the tangent and cotangent the diagram is slightly altered, these ratios being represented by lengths of lines on fixed tangents instead of variable ones. The student should draw the two kinds of diagrams for each of several angles lying in different quadrants and compare them, verifying that the sign and magnitude of the ratios agree with one another whichever diagram is used.

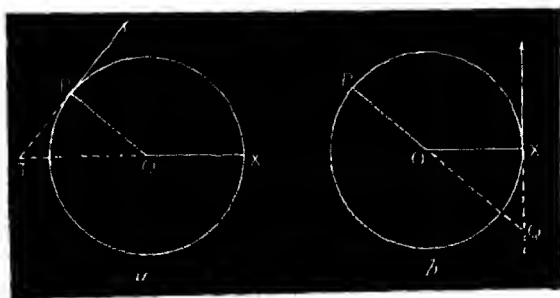


FIG. 90.

Thus, for the tangents of obtuse angles in Fig. 90 (a), the tangent PT, whose direction of measurement is shewn by the arrow-head, has to be produced **backward** to meet OX also produced backward: and in Fig. 90 (b) OP produced backward meets the tangent at X produced in the **negative direction**. Also the triangles OPT in (a) and

OXQ in (b) are congruent: hence the values of the tangents from the two diagrams are equal in magnitude and sign.

The graphs on pp. 173, 174, if accurately drawn, suggest nearly all the important relations between angles of any magnitude having equal or complementary ratios.

The method of construction at once shows a Periodicity of 2π for \sin , \cos , \sec , \csc ; and a Periodicity of π for the \tan and \cot . This can be further verified by taking a trace of any part of any of the graphs, and shifting the trace forward or backward through a distance corresponding to 2π parallel to the horizontal axis (i.e. the *length* of the page in the graphs given).

For the experiments which follow the student should draw (on tracing linen if possible) graphs of the ratios on as large a scale as is convenient. In order to obtain the true shapes of the graphs, $y = \sin x$, $y = \cos x$ etc., which is desirable, the radius of the circle must be so taken that

$$\frac{\text{radius of circle}}{\text{length representing } 90^\circ} = \frac{1}{\frac{\pi}{2}} = \frac{2 \times 113}{355}.$$

Thus in the graphs on pp. 173, 174, $\frac{1}{16}$ in. was taken to represent 10° , and hence the radius of the circle had to be

$$\begin{aligned} & \frac{9 \times 2 \times 113}{355} \text{ in.} \\ & = .57 \text{ in. nearly.} \end{aligned}$$

Each separate graph should first of all be examined for symmetry. The two kinds of symmetry usually considered are (1) symmetry about a line, (2) symmetry about a point.

Symmetry about a line exists when the figure can be folded about that line so that all points and lines of one part are brought into coincidence with corresponding points and lines of the other part (see p. 3).

Symmetry about a point (O) exists when to each point of the figure there corresponds another point so that the line joining them is bisected at O. In the case of symmetry round a point, one half the figure can be brought into coincidence with the other half, by rotation of the first half through an angle of two right angles: or part of the figure may be folded about any straight line through the pole of symmetry and again folded about a straight line through the pole at right angles to the first, and thus brought into coincidence with the other part. It should be observed that folding about the axis of x changes the signs of all the y 's; folding about the axis of y changes the signs of all the x 's; whilst folding about both axes (in case of symmetry with regard to a point) changes the signs of both x 's and y 's.

93. Consider the graph of $y = \sin x$, drawn true to scale and as large as possible, verify by folding (or by reference to the "circle of construction") that

(i) The graph is not symmetrical with regard to the axis of x , or any line parallel to it. Hence for each value of x there is only one value for the *sine*.

(ii) The curve is symmetrical with regard to the lines $x = (2n+1)\frac{\pi}{2}$, where n is any integer positive or negative.

Hence the values of the sines of angles equidistant from $(2n+1)\frac{\pi}{2}$ are equal; in particular $\sin A = \sin(\pi - A)$, these being equidistant from $\frac{\pi}{2}$.

(iii) The curve is symmetrical with regard to the pts. $x = n\pi$, $y = 0$ where n is an integer positive or negative.

Hence the values of the sines of angles equidistant from $n\pi$ are equal in magnitude but opposite in sign; in particular

$$\begin{aligned}\sin(-A) &= -\sin(+A), \\ \sin(\pi + A) &= -\sin(\pi - A) = -\sin A \text{ by (ii).}\end{aligned}$$

94. Consider the graphs for $\cos x$, $\tan x$ and deduce from the symmetry of these curves that

$$(1) \quad \cos(-A) = +\cos A,$$

$$(2) \quad \cos(\pi - A) = -\cos A,$$

$$(3) \quad \cos(\pi + A) = -\cos A,$$

$$(4) \quad \tan(-A) = -\tan A,$$

$$(5) \quad \tan(\pi - A) = -\tan A,$$

$$(6) \quad \tan(\pi + A) = +\tan A.$$

95. Shew that the graphs of $\cos x$ and $\sin x$ are identical in form, if the graph of the cosine is shifted forward a distance representing $x = \frac{\pi}{2}$. Hence deduce that for an angle of any magnitude whatever

$$\sin\left(\frac{\pi}{2} + A\right) = \cos A,$$

$$\begin{aligned} \text{hence} \quad \sin\left(\frac{\pi}{2} - A\right) &= \cos(-A) \\ &= \cos A \text{ by Expt. 94.} \end{aligned}$$

96. Superimpose the graphs of the tangent and cotangent and, observing that the graph for the cotangent is the image of the graph for the tangent in any one of the lines

$$x = (2n+1)\frac{\pi}{4},$$

deduce that

$$\cot\left(A + 2n+1\frac{\pi}{4}\right) = \tan\left(2n+1\frac{\pi}{4} - A\right)$$

and in particular

$$\cot A = \tan\left(\frac{\pi}{2} - A\right).$$

† 97. Find a formula for all the angles which have the same sine as a given angle θ .

From Expt. 93 we have

$$\sin A = \sin(\pi - \theta).$$

Hence, since the sine is a periodic function of period 2π ,

$$\therefore \sin A = \sin(2n\pi + \theta)$$

$$\sin(\pi - A) = \sin(2m+1\pi - \theta).$$

Now $(-1)^r$, where r is any integer, is equal to $+1$ or -1 according as r is even ($=2n$) or odd ($=2m+1$): hence the angles $2n\pi+\theta$, $\overline{2m+1}\pi-\theta$ are all included in the formula

$$r\pi + (-1)^r \theta.$$

$$\therefore \sin A = \sin(r\pi + (-1)^r \theta),$$

and the right-hand side of this equation includes *all* the values of the angle having the same sign as A , i.e.

$$\sin^{-1}(\sin \theta) = r\pi + (-1)^r \theta.$$

98. Verify the formula obtained for $\sin^{-1}(\sin A)$, by direct reference to a figure showing all these angles, and shew that there are no other values besides those given by the formula.

99. Shew, (i) by reference to the graph, (ii) by reference to a figure shewing all the angles, that the corresponding formulae for cosines and tangents are

$$\cos^{-1}(\cos \theta) = 2n\pi \pm \theta,$$

$$\tan^{-1}(\tan \theta) = n\pi + \theta.$$

EXERCISES.

1. Shew that

$$(a) \cos(180^\circ + A) = \cos(A - 180^\circ).$$

$$(b) \tan(90^\circ + A) = \cot(180^\circ - A).$$

$$(c) \operatorname{cosec}(180^\circ + A) = \operatorname{cosec}(-A).$$

2. Find an expression for all the angles satisfying the equation

$$\tan \theta = \cot \theta.$$

3. It can be shewn that

$$2 \sin \frac{\theta}{2} = \pm \sqrt{1 + \sin \theta} \pm \sqrt{1 - \sin \theta}.$$

Examine this formula to determine which of the ambiguous signs must be taken as θ increases from -2π to $+2\pi$.

4. An angle is known to lie between 540° and 630° : its sine is given $= -\frac{1}{2}$; find the sine and cosine of half the angle.

5. The tangent of an angle between 270° and 360° is $-\frac{3}{4}$: find its sine and cosine.

6. Shew that the signs of the ratios *sin*, *cos*, and *tan* are never all the same for any pair of quadrants; also that if any two of these ratios are given with their proper signs the magnitude of the angle ($\pm 2n\pi$) can be determined by a geometrical construction.

7. Find $\sin 37^\circ 20'$ from the tables, and calculate its cosine: hence find (a) $\sin 127^\circ 20'$; (b) $\cos 142^\circ 40'$; (c) the sine, cosine and tangent of $217^\circ 20'$, $307^\circ 20'$, showing the triangle OPM in a diagram for each case.

8. From the tables it is found that

$$\sin 21^\circ 20' = 0.36370 \text{ and } \cos 21^\circ 20' = 0.93148;$$

find A (less than four right angles) when

$$\sin A = -0.36379 \text{ and } \cos A = -0.93148;$$

also find B (less than four right angles) when

$$\sin B = 0.93148, \cos B = -0.36379.$$

9. Find the eight positive values of θ (less than four right angles) for which

$$2 \sin^2 2\theta = 1.$$

10. Examine the signs of the ratios *sin*, *cos*, and *tan*, for angles lying in the four quadrants, and tabulate the results.

| Quadrant | I | II | III | IV |
|----------|---|----|-----|----|
| sin | | | | |
| cos | | | | |
| tan | | | | |

11. Give verbal statements of the changes in sign and magnitude of (i) $\sin A$, (ii) $\cos A$, (iii) $\tan A$ as A increases from 0° to 360° .

12. *Draw a graph for

$$y = a \sin (vx + \beta)$$

where a , v , β are constants, and x is the circular measure of a variable angle.

* This is an important graph in problems on Sound, Cranks, and other branches of Higher Applied Mathematics.

TEST PAPERS (*continued*).

11.

[Board of Education, May 1903: Stage II., Trigonometry.]

1. (a) Define the logarithm of a number to a given base, and state what are the practical advantages of taking 10 as the base of a system of logarithms.

(b) Explain how it is that $\log(a^3) = 3 \log a$.

(c) Using the given table*, find $\log 36$, and the numerical value of the fifth root of 1.125. Find also the numerical value of $(\tan 35^\circ 40')^3$.

2. (a) Find from a diagram the numerical value of the sine of 60° , and of the tangent of 60° . Find from your answer the tabular logarithm of the sine of 60° .

(b) Given $\sin 21^\circ 20' = 0.3638$ and $\cos 21^\circ 20' = 0.9315$, find A (less than four right angles) when

$$\sin A = -0.3638 \text{ and } \cos A = -0.9315.$$

Also, find B (less than four right angles) when

$$\sin B = 0.9315 \text{ and } \cos B = -0.3638.$$

3. (a) If A is an acute angle, shew by means of a diagram:

(i) $\cos(90^\circ - A) = \sin A$,

(ii) $\cos(90^\circ + A) = -\sin A$,

(iii) $\tan(180^\circ + A) = \tan A$.

(b) Shew that:

(i) $2(1 + \sin A)(1 + \cos A) = (1 + \sin A + \cos A)^2$,

(ii) $\tan^2 A - \tan^2 B = \frac{\sec^2 B - \cos^2 A}{\cos^2 A \cos^2 B}$.

4. Shew that in any plane triangle the sides are proportional to the sines of the opposite angles. N.B.—Two cases are to be considered.

ABC is a right-angled triangle, and the line bisecting the right angle C cuts AB in D; shew that

$$DB = AD \tan A.$$

* The requisite logarithms are supplied with these examination papers.

5. The hypotenuse of a right-angled triangle is 1,000 feet long, and the difference between the other two sides is 240 feet ; calculate the other sides and angles of the triangle, and verify your result by drawing the triangle to scale.

6. P and Q are two stations 1,000 yards apart on a straight stretch of sea-shore, which bears East and West. At P a rock bears 42° West of South, at Q it bears 35° East of South. Shew that the distance of the rock from the shore is $1,000 \sin 48^\circ \sin 55^\circ \div \sin 77^\circ$ yards, and calculate this distance to the nearest yard.

12.

[Board of Education, May 1904 : Stage II., Trigonometry.]

1. Define the base of a system of logarithms.

State what number is taken as the base of the common logarithms.

Mention two properties of common logarithms which depend on the fact that that number is taken as their base.

Find by logarithms the numerical values of:

- (a) $(1.05)^{20}$;
- (b) $(\frac{2}{3})^{\frac{1}{2}}$;
- (c) $(\sin 18^\circ 42')^3$.

2. (a) Find, by the aid of a diagram, the sine, cosine, and tangent of 30° .

(b) Write down the values of $\cos 150^\circ$, $\tan 210^\circ$, $\sin 330^\circ$.

(c) If $\tan A = \frac{4}{3}$, find the true logarithm of $\tan A$, the tabular logarithm, and the number of degrees with the odd minutes and seconds in the angle A.

3. (a) The sine of an angle less than a right angle is 0.4; find the cosine, tangent, and cotangent of that angle.

(b) A is a positive angle less than four right angles, and it is given that

$$2 \sin A = \pm \sqrt{3};$$

find all the values of A.

(c) It is given that $\sin 32^\circ = 0.530$ and $\cos 32^\circ = 0.848$.

If $\cos B = 0.530$ and $\sin B = -0.848$, where B is a positive angle less than four right angles, explain why there can be only one value of B, and find it.

4. (a) If $\tan \theta + \cotan \theta = 3$, find $\tan \theta$.

(b) Find the value of

$$\sqrt{(\cotan^2 \theta - \cos^2 \theta)},$$

when

$$(a+b) \sin \theta = a-b.$$

5. (a) In any triangle ABC shew that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

(b) A, B, C are points on the circumference of a given circle; AD is drawn to meet the tangent at B at right angles in D, and AE is drawn to meet the tangent at C at right angles in E; shew that

$$\frac{AD}{AE} = \frac{AB^2}{AC^2}.$$

6. (a) In the triangle ABC, given $A=40^\circ$, $a=7$ units, $c=10$ units, find the two possible values of C and the corresponding values of B; and using the two values of B with the data given, find the two values of b.

(b) Construct the two different triangles ABC, in which $A=40^\circ$, $a=7$ units, $c=10$ units and write down the measure of the angle C in each case as read off from your protractor.

13.

[Board of Education, May 1905: Stage II, Trigonometry.]

1. (a) Define a logarithm.

Reasoning from your definition, find the logarithm of 81 to the base 3, and the logarithm of $\frac{1}{4}$ to base 2.

- (b) Find by means of the tables the common logarithms of

$$(i) 0.00144, \quad (ii) (\tan 34^\circ 47')^{\frac{1}{2}}.$$

- (c) Find the numerical value of

$$(6)^{\frac{1}{3}} \times (12)^{\frac{1}{3}} \div (125)^{\frac{1}{3}}.$$

2. (a) Find, with the aid of an appropriate diagram, the sine, cosine, and tangent of an angle of 30° , and also of an angle of 60° . Write down with proper signs the numerical values of the following expressions:

$$\sin 120^\circ, \quad \cos 150^\circ, \quad \tan 150^\circ.$$

(b) The angular elevation of the top of a spire is 60° , and the angular elevation of the top of the tower on which the spire stands is 30° ; the height of the tower is 50 feet; calculate the height of the spire, and verify your result by using a scale and protractor.

3. (a) The ratio of two geometrical magnitudes is commonly denoted by the letter π ; state what those magnitudes are.

Without going into much detail, indicate why you believe that π is nearly equal to 3.14159.

(b) Find the number of degrees, with the odd minutes and seconds, in the angle subtended at the centre of a circle by an arc twice as long as the radius.

4. Establish the formula

$$\left(\sin \frac{A}{2}\right)^2 = \frac{(s-b)(s-c)}{bc},$$

and by means of it find the greatest angle of the triangle whose sides are 13 feet, 30 feet, and 37 feet long respectively.

5. Find an expression for the area of a triangle (1) when two sides and the included angle are given, (2) when one side and the angles are given.

A and C are two points on a given straight line, and two parallel lines AB, CD are drawn on the same side of AC, X is a point in AC, Y a point in AB, Z a point in CD, shew that twice the area of the triangle XYZ equals

$$(CX \cdot AY + AX \cdot CZ) \sin BAC$$

6. ABC is a triangle having a right angle at A, let D be the middle point of BC, and E the point in which the line bisecting the right angle cuts BC, shew that

$$2DE = BC \tan (45^\circ - B).$$

If we suppose that a circle is described about the triangle ABC, explain how the point E moves along BC, when A moves from B to C along the circumference of the circle.

14.

[Board of Education, May 1906 Stage II, Trigonometry]

1. Explain why the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers

By means of logarithms given below, find the fifth root, and the fifth power of 0.69889 correct to five decimal places

If $4 \log_{10} x + 7 = 0$, find x

If $y \log_{10} 0.0424 = \log_{10} 0.2165$, find y to four places of decimals

2. Draw an appropriate diagram, and from it find the numerical values of the sine, cosine, and tangent of an angle of 45°

Find also the true logarithm and the tabular logarithm of $\sin 45^\circ$ and of $\tan 45^\circ$.

If A be any angle between 0° and 90° , find $\tan A$ in terms of $\sin A$, and also in terms of $\cos A$

3. Shew, in a carefully drawn diagram, an angle 234° , and explain, with reference to your diagram, why both the sine and the cosine of that angle are negative.

Assuming that $\sin 36^\circ 56'$ is 0.6, and that $\cos 36^\circ 56'$ is 0.8, find the angle whose sine is -0.6 and whose cosine is +0.8.

Find, from the annexed table, the numerical value of $\sin 326^\circ 42'$.

4. Establish the following identities:—

$$(a) \quad \sin^4 A + \cos^2 A = \cos^4 A + \sin^2 A.$$

$$(b) \quad \sin^2 A \tan^2 A = \tan^2 A - \sin^2 A.$$

$$(c) \quad \sin A \cos A = \frac{\tan A}{1 + \tan^2 A}.$$

$$(d) \quad \frac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B} = \frac{1}{\tan^2 A \tan^2 B} - 1.$$

5. Two points A and B are 2,000 yards apart on a straight road, and P is a flagstaff off the road; it is found that the angles PAB and PBA are $33^\circ 18'$ and $105^\circ 20'$ respectively.

Calculate the distance BP, and the number of square yards in the triangle ABP.

6. Shew that the area of a quadrilateral is equal to the area of a triangle having two sides equal to the diagonals of the quadrilateral, and the included angle equal to either of the angles between the diagonals.

Find the area of the quadrilateral in which the diagonals are 216.5 ft. and 447.5 ft. long respectively, and are inclined to each other at an angle of $116^\circ 30'$.

ANSWERS.

PAGE

- 12.** 2. 120° . 3. (i) $\frac{1}{2}^\circ$, (ii) $6'$. 4. (i) $17\frac{1}{2}^\circ$, (ii) 70° , (iii) $129\frac{1}{2}^\circ$.
7. $16^\circ 40' 0''$. 8. 21. 9. $43^\circ 19' 51''$, $86^\circ 39' 42''$.
- 13.** 1. $11\frac{1}{4}^\circ$. 2. $56\frac{1}{4}^\circ$, $56\frac{1}{4}^\circ$. 3. $B=119^\circ$, $C=117^\circ$, $D=84^\circ$,
 $E=159^\circ$, $F=61^\circ$. [Approx.]
- 14.** 5. C, $2\frac{1}{4}$ miles, S. by 19° W.; D, 5 miles, W. by 31° S.;
E, $4\frac{1}{2}$ miles, W. by 15° N.; F, $5\frac{1}{4}$ miles, N. by 42° W. [Approx.]
6. $13\frac{3}{4}$ miles. [Approx.] 7. 21 knots, S. by 37° E.
8. 108 yds.
- 21.** 1. 3 ft. $6\frac{1}{2}$ in. 2. $55\frac{1}{2}^\circ$. 3. 117 ft.
- 24.** 4. $28\frac{1}{2}$ ft. 5. 1219 ft., 361 ft. 6. 132 ft., 119 ft.
Expt. 19. 212 ft.
- 25.** 7. 639 yds. 8. $5\frac{1}{2}^\circ$, $16\frac{1}{2}^\circ$. 9. 0.12 in., 2.65 in. 10. 67 ft.
11. 14 ft. 12. 50 ft. 9 in. 13. 127 ft. 14. 147 ft.
- 42.** 1. Let ABCD be \square^m , $AB=15$, $BC=24$. Draw AE and DF
 $\perp BC$. Find, in succession, AE, BE, CE, $\angle ACB$, $\angle DBF$;
hence $AC=19.2$, $BD=33$. 2. 65 ft. 3. 28.2 sq. feet.
4. 125 ft. 5. 280 ft. 6. $891\frac{1}{2}$ ft.
- 43.** 7. 154 ft. 8. $11^\circ 5'$. 9. 2329 yds. 10. $87\frac{1}{2}$ seconds.
11. 1383 ft. 12. 116 ft.

| | If | $\sin A$ | $\cos A$ | $\tan A$ | $\cot A$ | $\sec A$ | $\operatorname{cosec} A$ |
|------------------------------|----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $\sin A = s$ | | s | $\sqrt{1-s^2}$ | $\frac{s}{\sqrt{1-s^2}}$ | $\frac{\sqrt{1-s^2}}{s}$ | $\frac{1}{\sqrt{1-s^2}}$ | $\frac{1}{s}$ |
| $\cos A = c$ | | $\sqrt{1-c^2}$ | c | $\frac{\sqrt{1-c^2}}{c}$ | $\frac{c}{\sqrt{1-c^2}}$ | $\frac{1}{c}$ | $\frac{1}{\sqrt{1-c^2}}$ |
| $\tan A = t$ | | $\frac{t}{\sqrt{1+t^2}}$ | $\frac{1}{\sqrt{1+t^2}}$ | t | $\frac{1}{t}$ | $\sqrt{1+t^2}$ | $\frac{\sqrt{1+t^2}}{t}$ |
| $\cot A = x$ | | $\frac{1}{\sqrt{1+x^2}}$ | $\frac{x}{\sqrt{1+x^2}}$ | $\frac{1}{x}$ | x | $\frac{\sqrt{1+x^2}}{x}$ | $\sqrt{1+x^2}$ |
| $\sec A = y$ | | $\frac{\sqrt{y^2-1}}{y}$ | $\frac{1}{y}$ | $\frac{\sqrt{y^2-1}}{1}$ | $\frac{y}{\sqrt{y^2-1}}$ | y | $\frac{y}{\sqrt{y^2-1}}$ |
| $\operatorname{cosec} A = z$ | | $\frac{1}{z}$ | $\frac{\sqrt{z^2-1}}{z}$ | $\frac{1}{\sqrt{z^2-1}}$ | $\sqrt{z^2-1}$ | $\frac{z}{\sqrt{z^2-1}}$ | z |

48. 1. $\frac{3}{4}, \frac{5}{4}$. 2. $\frac{2\sqrt{2}}{3}, \frac{1}{2\sqrt{2}}$. 3. $\frac{4}{5}, \frac{5}{4}$. 4. $\frac{1}{\sqrt{15}}, \frac{\sqrt{15}}{4}$.
 5. $\frac{\sqrt{3}}{2}, \frac{1}{2}$. 6. $\frac{\sqrt{5}}{3}, \frac{3}{2}$. 7. $\frac{b}{\sqrt{c^2-b^2}}$. 8. $\frac{a}{\sqrt{a^2+b^2}}$,
 $\frac{b}{\sqrt{a^2+b^2}}$. 9. $\frac{\sqrt{a^2-1}}{a}, \frac{1}{\sqrt{a^2-1}}$. 11. $k^2(1+k^2)=1$.

59. 1. (a) $\sin=0.19156$ [0.1915562]*, $\sin=0.63415$ [0.6341533],
 $\tan=0.19517$ [0.1951703], $\tan=0.82016$ [0.8201597].
 (b) $\cos=0.56712$ [0.5671212], $\cos=0.13702$ [0.1370220],
 $\operatorname{cosec}=1.21413$ [1.2141310], $\operatorname{cosec}=1.00952$ [1.0095217].
 (c) 1.00000 [1.0000003], 1.05994 [1.0599366].
 (d) 0.54660 [0.5466022], 0.19044 [0.1904387].
 2. (a) $30^\circ, 60^\circ, 15^\circ, 45^\circ, 18^\circ$.
 (b) $14^\circ 28' 39''$ [14° 28' 39''], $19^\circ 28' 15''$ [19° 28' 16''],
 $77^\circ 27' 47''$ [77° 27' 49''], $35^\circ 15' 50''$ [35° 15' 52''].
 (c) $71^\circ 33' 53''$ [71° 33' 54''], 30° [30°], 60° [60°],
 $19^\circ 28' 17''$ [19° 28' 16''].

* The numbers in brackets have been obtained from seven figure tables, and are inserted for the sake of comparison with the results obtained from the five figure tables on pages 36—41.]

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63. 7. (1) 210; (2) 1872; (3) 156, $\cos C = -\frac{9}{13} \therefore C$ is obtuse.

$$\begin{aligned} \mathbf{70.} \quad 2. \quad \sin 15^\circ &= \frac{\sqrt{3}-1}{2\sqrt{2}}, & \sin 75^\circ &= \frac{\sqrt{3}+1}{2\sqrt{2}}, \\ \cos 15^\circ &= \frac{\sqrt{3}+1}{2\sqrt{2}}, & \cos 75^\circ &= \frac{\sqrt{3}-1}{2\sqrt{2}}, \\ \tan 15^\circ &= \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}; & \tan 75^\circ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}. \end{aligned}$$

71. 4. $45^\circ, 60^\circ, 75^\circ$. 5. 120° . 6. $A=54^\circ$ or 126° , $B=108^\circ$ or 36° , $b=\sqrt{10+2\sqrt{5}}$. 7. $a=2(\sqrt{3}-1)$, $c=2\sqrt{2}$.
 8. $50\sqrt{3}$. 9. $C=75^\circ$ or 105° , $A=90^\circ$ or 60° , $a=2\sqrt{2}$ or $\sqrt{6}$.
 10. $5\sqrt{3}:8:4+3\sqrt{3}$; the sine of the third angle is found by the formula in Ex. 5, p. 63.
 11. $47^\circ 41' 15''$, $38^\circ 56' 32''$: hence the third angle is $93^\circ 22' 23''$, although the calculated value is $93^\circ 22' 22''$; see p. 97.
 12. 169.7. 13. 7.49 in., 8.58 in.

72. 14. 74.6 ft. 15. $42^\circ 19' 4''$. 16. $AM=23,400$ yds., $BM=23,280$ yds. 17. 10.6. 18. 226 ft. 20. 6000 ft.

77. 1. (a) 1.66; (b) 0.3; (c) 2.35; (d) 4.71. 2. 1.732; 2.236; 2.45; 2.644. 3. 1.26; 1.587; 2.08.

79. 2. 0.60206; 0.77815; 0.90309; 1.07918; 1.20412; 1.25527.
 4. 1.43136; 1.50515; 1.80618; 1.90849. 6. 0.15052; 0.15904; 0.31808; 0.53959. 9. 0.69897; 1.17609; 0.17609; 0.87506; 0.09691; 0.55630; 0.68124.

80. 10. 0, 1, 2, 3. 11. 2.09684 by interpolation: 2.09691 by factors: hence the ratio $\frac{126-120}{120}$ is too large to allow the rule of proportional differences to give a trustworthy figure in the fifth decimal place. It will be found on reference to the tables that $2.09684 = \log_{10} 124.98$ instead of $\log_{10} 125$.
 12. (a) 2.00328. (b) 2.00329: seven-figure tables give 2.0032882. 13. By interpolation we get 4.39946 instead of 4.39947 from the factors: seven-figure tables give 4.3994660.

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81. 14. $1.53466, \bar{2}.53466$.**82.** 15. $3.672, 0; 2.3075 \times 10^2, 2; 2.3 \times 10^{-2}, \bar{2}; 3 \times 10^{-1}, \bar{1}; 1 \times 10^{-5}, \bar{5}$. 16. The powers of ten appearing as factors when the numbers are reduced to standard form are respectively $10^0, 10^2, 10^{-1}, 10^0$.**86.** 5. 0.33491. 6. 0.1118. 7. 428270. 8. 0.076539.
9. 0.65072. 10. 2.6534. 11. 0.80911. 12. 0.00011708.
13. 59.6. 14. 0.99178. 15. $\bar{1}.56941$. 16. $\bar{1}.79022$.
17. 0.01223. 18. 1.1063. 19. 1.6107.**97.** 3. (i) 8.5889226; 9.0507984; 9.9923305; 9.9996048.
(ii) 10.2449297; 10.7327623; 10.0296687; 11.0918799.
(iii) 10.0870962; 10.0037598; 10.2793620; 10.9221373.**98.** 4. $2.0024685; 3.0001303; \bar{1}.0093488; 2.0159965; \bar{2}.0345321;$
 $0.0000087; 3.0409041$.**99.** 5. (1) 2.9541074; 2.9506569; $\bar{1}.8594127; 0.9924117$.
(2) 971.941; 8921.748; .899477.**108.** 4. $r=4, r_1=24, r_2=12, r_3=8, R=10$.

5. (i) $38^\circ 52' 48''; 67^\circ 22' 51''; 73^\circ 44' 25''; 204.$
 $[38^\circ 52' 48.2''] [67^\circ 22' 48.5''] [73^\circ 44' 23.3'']$
- (ii) $112^\circ 37' 9''; 36^\circ 52' 13''; 30^\circ 30' 39''; 66.$
 $[112^\circ 37' 11.5''] [36^\circ 52' 11.6''] [30^\circ 30' 36.9'']$
- (iii) $143^\circ 7' 47''; 20^\circ 36' 36''; 16^\circ 15' 38''; 462.$
 $[143^\circ 7' 48.4''] [20^\circ 36' 34.9''] [16^\circ 15' 36.7'']$
- (iv) $75^\circ 45' 3''; 67^\circ 22' 51''; 36^\circ 52' 13''; 126.$
 $[75^\circ 44' 59.9''] [67^\circ 22' 48.5''] [36^\circ 52' 11.6'']$

N.B. The above results should be carefully compared in connection with the notes on pp. 110, 111.

5. 0.017 sq. in. 6. $26^\circ 33' 54''; 116^\circ 33' 54''; 36^\circ 52' 12''$.**117.** 6. 23 ac. 2 rd. 16 pl. 7. $74^\circ 58' 38''$. 8. $B=98^\circ 3' 53''$,
 $C=30^\circ 56' 7''$. 9. $B=45^\circ 23' 28''$, $C=99^\circ 0' 12''$, $c=3004.2$.
10. (i) $9^\circ 57' 37''$, $151^\circ 23' 23''$, 32.62 .
(ii) $132^\circ 46' 23''$, $28^\circ 36' 37''$, 137.96 .

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- 118.** 11. (i) Included angle = $41^{\circ} 47' 57''$, remaining side = 201.4 ft
 (ii) Included angle = $138^{\circ} 12' 3''$, remaining side = 468.4 ft
 12. 8578 yds 13. 47.7 ft, 63.9 ft. 14. 317.6 ft
 15. 134.8 ft 16. (1) 5355 ft, (2) $6^{\circ} 10'$, (3) $48^{\circ} 4' 51''$,
 (4) 16929 yds
120. 1. 29 462 in, 748.3 mm
126. 3. Divide the scale into degrees and thirds take arc on vernier equal to 13° and divide into 40 equal parts
 4. Radius of arc about 12 in, divided into intervals of $3'$, 179 of these divisions are divided, on the vernier, into 180 equal parts

TEST PAPERS

N B *Seven figure tables have been used in the calculation of the answers to these test papers*

- 138.** 1. 1 (a) $70\frac{5}{8}$, (b) 81° 2. $22\frac{1}{2}^{\circ}$ 3. 12.52 p.m.
 4. 10 in very nearly
139. 2. 1. 11 miles nearly 2. $8\frac{3}{8}$, $8\frac{3}{8}$, $5\frac{1}{8}$
 3. (i) $2\frac{1}{2}$, (ii) 90° , (iii) 1, (iv) $\sqrt{1-c^2}$, $\sqrt{1-s^2}$,
 $\frac{s}{\sqrt{1-s^2}}$, $\frac{t}{\sqrt{1+t^2}}$, $\frac{1}{\sqrt{1+t^2}}$, $\frac{\sqrt{1-c^2}}{e}$
 3. 2. $\sin \theta = \frac{4}{5}$, $\cos \theta = \frac{3}{5}$, $\theta = 53^{\circ} 7' 48''$
 3. $\sin \theta = \frac{c(b-a)}{a^2+b^2}$, $\cos \theta = \frac{c(b+a)}{a^2+b^2}$ 4. $50^{\circ} 36'$
140. 4. 1. 225 ft. Draw a vertical line TF, and a horizontal line TH. With the protractor set off TX, TY, TZ making angles 34° , 36° , 40° with the horizontal TH; join XY, produce XY to W, making YW=XY, draw WV||YT to meet TZ in V, join XUV and draw ABC||XUV, so that AB=BC=4". Through A draw AF||HT. Then TF gives the height required on a scale of 1"=25 ft.
 4. $4\ 12''$, $51^{\circ} 41' 2''$
 5. 1. $7\frac{1}{2}$ inches 3. (i) $\cos \theta = \frac{1}{2}$, $\theta = 60^{\circ}$,
 (ii) $\tan \theta = \frac{1}{2}$ or $\frac{3}{4}$, $\theta = 56^{\circ} 18' 35''$ or $36^{\circ} 52' 12''$
141. 5. Velocity
 6. 1. 10.58 in, 26.61 in, 56.54 sq in 2. (i) $1\frac{2}{3}$,
 (ii) 2.291 3. 297.35 sq in, 309.49 sq in;
 312.01 sq in 4. -1 5. 14.36 ft

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- 142.** 7. 1. $\frac{1}{10\cdot5}$, $\frac{1}{42\cdot2}$, $\frac{1}{151}$ up; $\frac{1}{66}$, $\frac{1}{110}$ down; $\frac{1}{251}$,
 $\frac{1}{143}$, $\frac{1}{48}$, $\frac{1}{48}$, $\frac{1}{132}$, $\frac{1}{293}$, $\frac{1}{121}$, $\frac{1}{406}$ up; $\frac{1}{27\cdot3}$,
 $\frac{1}{39\cdot6}$, $\frac{1}{17\cdot4}$, $\frac{1}{278}$ down; $\frac{1}{1760}$, $\frac{1}{660}$ up; $\frac{1}{13\cdot4}$ down.

N.B. *The technical term "gradient" as used by railway men indicates the sine of the angle of slope, and not the tangent.*

3. 11·716 miles. 4. $122\frac{1}{2}^\circ$ about. 5. $30^\circ 23' 49''$.
8. 2. 0·8571736, 3·499669, 21·30085.
143. 4. -1, -1, $\frac{1}{4}$, $\frac{3}{8}$, 3·32193.
 9. 1. 1391'96. 2. $\log_{10} e = 0\cdot4342942$. 3. 179266
 sq. ft. 4. '005 % nearly. 5. $24^\circ 25'$, $45^\circ 35'$.

- 144.** 10. 2. $\log 5 = 2a - c$, $\log 7 = c - a$, $\log 13 = b + 2c - 4a$.
 4. $20^\circ 37'$, $89^\circ 51'$, 66·518 ft.; error in third side is less than $\frac{1}{100}$ in. 5. (a) 59 scale divisions divided into 60 equal parts on vernier; (b) zero of vernier between 20·2 and 20·3 of scale and division 6 of vernier opposite 20·8 on scale; where 9 scale divisions equal 10 vernier divisions.

THEORETICAL.

- 157.** 1. $143^\circ 14' 22''$. 2. 0·63274. 3. 3 ft. 4. 67 in.
 5. 469 yds. 1 ft. 6. 24 minutes.
160. 1. Slightly more than 224. 2. $\frac{3\pi}{5}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{4\pi}{5}$.
 3. 57 in. 4. 62 in. 5. 3 : 8. 6. 6.
 7. Area = $\frac{1}{4\pi} \times (\text{circumference})^2$; 45·7 in. 8. 4 ft.
 nearly. 9. 5·77 in.; 8·16 in. 10. £24. 1s. 3d.
171. Expt. 92. (1) - 0·86603; (2) + 1; (3) - 0·70711;
 (4) + 0·86603; (5) - 1; (6) infinity.
179. 2. $(2n+1)\frac{\pi}{4}$. 4. -5, +0·86603. 5. -0·75, +0·8.
 7. (a) 0·79512; (b) 0·60645; (c) -0·60645, -0·79512,
 +0·76272; -0·79512, +0·60645, -1·64894. 8. $201^\circ 20'$,
 $111^\circ 20'$. 9. $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, $\frac{7\pi}{8}$, $\frac{9\pi}{8}$, $\frac{11\pi}{8}$, $\frac{13\pi}{8}$, $\frac{15\pi}{8}$.

TEST PAPERS (*continued*).(N.B. *Answers calculated with seven-figure tables.*)

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- 181.** 11. 1. (c) 1.5563025, 1.023835, 0.369669. 2. (a) 0.8660254, 1.7320508, 9.9375306; (b) See Ans. Ex. 8, p. 180.
- 182.** 5. 576.85, 816.85, 35° 13' 45.4", 54° 46' 14.6".
6. 625 yds.
- 12.** 1. (a) 26.533; (b) 0.8091064; (c) 0.0329415.
2. (a) $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$; (b) $-\frac{\sqrt{3}}{2}$, $+\frac{1}{\sqrt{3}}$, $-\frac{1}{2}$;
(c) 1.7569620, 9.7569620, 29° 44' 42".
- 183.** 3. (a) 0.9165150, 0.4364361, 2.2912858; (b) 60°, 120°, 240°, 300°; (c) B = 202°.
4. (a)* $\tan \theta = \frac{3 \pm \sqrt{13}}{2} = 3.3027756$ or 0.3027756;
(b) $\frac{4ab}{a^2 - b^2}$. 6. (a) C = 66° 40' 28" or 113° 19' 32",
B = 73° 19' 32" or 26° 40' 28", b = 10.535 or 4.889.
- 184.** 13. 1. (c) 4, -2; (b) 3.1583625, 0.893814; (c) 0.89329.
2. (a) $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$; $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$; $+\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\frac{1}{\sqrt{3}}$;
(b) 100 ft. 3. (b) 114° 35' 30". 4. 112° 37' 14".
- 185.** 6. A graph of $DE = \frac{1}{2} \tan \left(\frac{\theta}{2} - 45^\circ \right)$ where θ is the angle BDA will show the displacement; two derived curves for the velocity and acceleration should be drawn by plotting the "rate of increase" in each curve against θ to obtain the respective derived curves.
- 14.** 1. 0.93085; 0.16674; 0.0177828; 0.4841.
- 186.** 3. 323° 4', -0.5490228. 5. $1153\frac{1}{2}$ yds., 1112201 sq. yds. 6. 433523 sq. ft

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